

## Exploring University Mexican Students' Quality of Intra-Mathematical Connections When Solving Tasks About Derivative Concept

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### Abstract

The quality of the intra-mathematical connections made by Mexican university students when solving tasks on the derivative was characterized. The typology of mathematical connections and the quality levels of mathematical connections were used. Interviews were conducted based on eight tasks applied to three case studies. The results showed that the quality of the intra-mathematical connections: different representations, procedural, implication, part-whole and meaning of the students, is found mostly at level 2 (consistent and argued mathematical connections) and only one student presented inconsistencies to find the correctness equation of the tangent line to the curve at a point (quality level 0 connection). Likewise, the metaphorical connection was identified when a student mentioned the metaphorical expression "*the graph has no holes*", which suggests the conceptual metaphor "*the graph is a path*" referring to the continuity of a function. We believe that quality level 2 mathematical connections ensure understanding, and the quality level 0 mathematical connections are the reasons why students have difficulty understanding the derivative.

**Keywords:** quality of mathematical connections, derivative, university students, thematic analysis

### INTRODUCTION

Mathematical understanding is achieved through the ability to establish mathematical connections, which allow mathematics to be seen as an integrated field (García-García, 2019; National Council of Teachers of Mathematics [NCTM], 2000). If a student connects mathematical ideas, meanings, representations among themselves, these with other mathematical concepts and with those of other disciplines, his or her mathematical understanding will be deeper and he or she will solve mathematical problems in a consistent way (Adu-Gyamfi, Bossé & Chandler, 2017; Breda, Hummes, Da Silva & Sánchez, 2021; Da Fonseca & Henriques, 2020; Dolores-Flores, Rivera-López & García-García, 2019; Evitts, 2004; García-García & Dolores-Flores, 2021a, 2021b; NCTM, 2000; Rodríguez-Nieto, 2020; 2021; Rodríguez-Nieto, Rodríguez-Vásquez & García-García, 2021).

Currently, in several research, the study of mathematical connections has been of interest, for example, Lyublinskaya (2006) argues that the practical

applications of different sciences such as chemistry and physics contribute to the understanding of students and teachers about algebra and more when making connections with real situations. In this sense, activities of scientific phenomena were connected with quadratic and trigonometric functions modeled and verified with a CBL2™ calculator. Aguilar-González et al. (2018) were interested in understanding the professional knowledge of a teacher in the fifth grade of primary education, emphasizing the relationships or connections between the subdomains of the Mathematics Teacher Specialized Knowledge (MTSK) model and, Conceptions on Teaching and Learning Mathematics, because seeing beliefs and conceptions in a disjointed way of knowledge results in an incomplete image of the teacher and their mathematical practice. Specifically in this research, the connections were recognized when the conceptions are related to one or more subdomains, which are activated through the evidence of a conception in the episodes where the teacher's knowledge is manifested.

In the research of Breda, Pino-Fan and Font (2017), Breda, Font and Pino-Fan (2018) and Breda (2020), some

### Contribution to the literature

- The quality of the mathematical connections established by university students is reported when they solve problems on the derivative in different contexts: verbal, numeric and graphic.
- This research shows an example of the quality of metaphorical connections established by students, which had been activated in other studies by in-service teachers.
- We consider that the quality of mathematical connections influences students' difficulties in solving problems on the derivative and their understanding.

components and indicators of the criteria of epistemic and ecological suitability are considered where the connections are important. For example, in epistemic suitability, two components are taken into account: 1) *richness of processes* that refers to the relevant processes in mathematical activity (e.g., modeling, problem solving, argumentation, connections) that are part of the sequence of tasks, and 2) *representativeness of the complexity* that involves the partial meanings (e.g., definitions, properties, procedures, etc.), as representative samples of the complexity of the mathematical notion chosen to be taught as part of the curriculum. According to Breda et al. (2017) for one or more partial meanings, a representative sample of problems and the use of forms of expression (e.g., verbal, graphic, gestural, symbolic) are provided, with their respective treatments and conversations between them. Likewise, ecological suitability takes into account the component of intra and interdisciplinary connections, referring to the fact that the content taught is related to other mathematical topics.

For his part, Amaya (2020) reported that pre-service mathematics teachers recognize the properties of functions but find it difficult to connect different representations and relate the functions to the sociocultural context. He found that the lack of rich and connected relationships between representations hinders pre-service teachers from converting between different representations of a function. In fact, the consideration of domain values as discrete was an aspect that prevented establishing good connections between representations, which would allow assigning meaning and meaning to the mathematical function object. In another research, Dolores-Flores and Ibáñez-Dolores (2020) affirm that, "the understanding of the slope requires the formation of internal networks as a product of connections between conceptualizations in the intra and extra-mathematical plane, in addition to the harmonious development of conceptual knowledge and procedural" (p. 825). Campo-Meneses, Font, García-García and Sánchez (2021) identified that, "the connection of reversibility is essential for achieving students' full understanding of the existent relationship between the exponential and logarithmic function; however, this requires a network of connections" (p. 1).

In relation to the concepts of Calculus, Vargas et al. (2020) analyzed first year high school textbooks from

editorials Anaya, Bruño, Edelvives, Santillana and SM, emphasizing the derivative issue, where they found three types of meanings of the derivative: the *first* it refers to the procedural-algebraic meaning without connections based on rules of derivation of simple functions. The *second* is an algorithm-based meaning similar to the *first*, but it is aimed at the derivation of compound functions, and the *third* meaning is the geometric conceptual one based on connections between algebraic, verbal and symbolic representations.

Dolores-Flores and García-García (2017) investigated the intra-mathematical and extra-mathematical connections, García-García and Dolores-Flores (2018, 2021a, 2021b) emphasized the mathematical connections established by pre-university students when they solve tasks that involve the derivative and the integral. There are other research that show students' difficulties in learning the derivative concept, reporting that they mostly use derivation rules and solve problems mechanically through formulas (Fuentealba et al., 2015, 2018a, 2018b; Muzangwa & Chifamba, 2012; Sánchez-Matamoros et al., 2008). Furthermore, Feudel and Biehler (2020) argue that students have insufficient understanding of the concept of rate, which is one of the reasons why students have difficulty understanding the derivative in application contexts.

Rodríguez-Nieto et al. (2021) analyzed the mathematical connections established by pre-service teachers and found that some of them have difficulties in finding the equation of the tangent line because they confuse the derivative function with the tangent line, which is caused by the inappropriate meaning they have about the derivative in a point. Regarding the mathematical connection between graphical and analytical representations, students presented difficulties in sketching the graph of derivative  $f'$  from the graph of  $f$  (Berry & Nyman, 2003; Ferrini-Mundy & Graham, 1994; García-García & Dolores-Flores, 2021a). Asiala et al. (1997) reported that students had difficulties in representing the graph of the derivative, because they related little the graphical, numerical and analytical forms, for example, they had difficulties to interpret the tangent line. In this sense, students set aside the geometric interpretation of the derivative and prefer the algebraic method (Zandieh & Knapp, 2006). In fact, in Rodríguez-Nieto, Rodríguez-Vásquez, Font and Morales-Carballo (2021) it was reported that a student

has a good understanding of the derivative when solving tasks based on the establishment of mathematical connections made up of a conglomerate of practices, processes, objects and semiotic functions that relate them, and, they recognized that if a student does not make connections that are necessary to solve a task, it is the origin of the cause of the difficulties to understand the derivative.

Hashemi et al. (2014) reported that students have limited conceptual understanding of the derivative, due to the poor connection between algebraic and graphical aspects. Likewise, students have difficulty connecting multiple representations of the derivative (Sari et al., 2018). Borji et al. (2018) reported that the majority of the students stayed at the inter level since it is difficult for them to relate the function  $f$  with its derivative and graph the derivative  $f'$  similar to the function  $f$ . Other students did not associate the relationship between the concavity of  $f$  with the growth or decrease of  $f'$ . Pino-Fan et al. (2017) reported a case study where a student presents difficulties in connecting symbolic and graphical representations of the absolute value function and deficiencies to work derivative from said function. Likewise, Pino-Fan et al. (2015, 2018) mentioned that pre-service teachers have difficulties connecting partial meanings of the derivative.

In Fuentealba et al. (2018a) and Fuentealba et al. (2018b) students had difficulties in: relating both the monotony and the curvature of  $f$  and the sign of  $f'$  and  $f''$ , as well as establishing bidirectional relationships or reversal of processes where the signs of  $f'$  and  $f''$  with  $f$  are linked; to deal with graphical information and the punctual analytical meaning of the derivative; to establish relationships of logical equivalence, because they build direct relationships or conjunctions from analytical information and find it difficult to determine extreme values and inflection points from the graphical information. Furthermore, in Indonesia, students have difficulties in solving derivative problems whose answers are not directly related to a formula, since they tend to memorize them. This implies that meanings cannot be connected in the definition and in the formula, in addition, students lack mathematical connections and imagine mathematical concepts in other areas (Nurwahyu et al., 2020).

From the literature, we identified that the research mostly focus on the difficulties that students manifest in solving tasks that involve the derivative or in exploring the mathematical understanding that they achieve from different theoretical perspectives. However, we consider, like Hiebert and Carpenter (1992), and Barmby et al. (2009) that a student understands a mathematical concept depending on the strength and quality of mathematical connections and not the number of connections. In this sense, several studies suggest that, the quality of mathematical connections made by students and teachers should be investigated, and how

the connections influence the understanding of mathematical concepts (Mhlolo, 2012; Mhlolo et al., 2012; García-García & Dolores-Flores, 2021a, 2021b; Rodríguez-Nieto, Font, Borji & Rodríguez-Vásquez, 2021; Rodríguez-Nieto, Rodríguez-Vásquez & Font, 2020; Rodríguez-Nieto et al., 2021). Therefore, in this research we answer the question: *What is the quality of the mathematical connections made by mexican university students when solving tasks that involve the derivative concept?*

## CONCEPTUAL FRAMEWORK

### Mathematical Connections in Mathematics Education

In Hiebert and Carpenter (1992), mathematical understanding is given by a network of internal representations of mathematical ideas, procedures and mathematical facts. In this sense, achieving mathematical understanding requires students' ability to establish mathematical connections (Hiebert & Carpenter, 1992; Eli et al., 2013). For the purposes of this research, mathematical connections were taken as "a cognitive process through which a person relates two or more ideas, concepts, definitions, theorems, procedures, representations, and meanings with each other, with other disciplines or with real life" (García-García & Dolores-Flores, 2018, p. 229). In addition, according to Businskas (2008), Mhlolo (2012), Dolores-Flores and García-García (2017), García-García and Dolores-Flores (2018, 2021a), mathematical connections are classified in intra-mathematical which "are established between concepts, procedures, theorems, arguments and mathematical representations of each other" (Dolores-Flores & García-García, 2017, p. 160), and extra-mathematical connections which "establishes a relationship of a mathematical concept or model with a problem in context (not mathematical) or vice versa" (Dolores-Flores & García-García, 2017, p. 161). However, in this research we only considered the intra-mathematical connections (García-García & Dolores-Flores, 2018, 2021a, 2021b) with the version of the Extended Theory of Connections (ETC) as reported in Rodríguez-Nieto et al. (2020) (see Table 1).

### Quality of Mathematical Connections

To analyze the quality levels of the mathematical connections (L), we adapt the analytical tool on the levels of quality of knowledge proposed by Mhlolo (2012), and Mhlolo et al. (2012). In this sense, a mathematical connection has been encoded as a *level 0* when there is not a mathematical connection on the student's answer; is *level 1* when a student makes mathematical connections, but he or she does not argue their answers, and is *level 2* if a student made mathematical connections and consistently justifies his or her answers. Specifically, level 2 of the quality of mathematical connections is reached when the subject makes argued connections, for

**Table 1.** Types of intra-mathematical connections

1) Different representations (DR)
These connections can be alternate or equivalent (Businskas, 2008). It is alternate if a student represents a mathematical concept in two or more different ways in different registers of representation: graphical-algebraic, verbal-graphical, etc. For example, an alternate representation of $y = mx + b$ is a straight line geometrically. The equivalent representation is a transformation within the same register (algebraic-algebraic, graphical-graphical, etc.). For example, $f(x) = x^2 + 5x + 6$ is equivalent to $f(x) = (x + 3)(x + 2)$ in the algebraic register.
2) Part-whole (PW)
This connection occurs when someone identifies that A is a generalization of B, where B is a particular case of A. For example, the function $g(x) = x^3 - 6x^2 + 12x - 3$ is a particular case of $f(x) = ax^3 + bx^2 + cx + d$ (Businskas, 2008).
3) Implication (I)
These connections are of the form A implies B as a logical relationship (Businskas, 2008; Mhlolo, 2012). For example, for the existence of derivative of a function $f$ at a point $P(x_0, y_0)$ , the function $f$ must be continuous at point P.
4) Procedural (P)
These connections are of the form, A is a procedure used when someone is working with B. For example, if a student has the slope $m$ and a point $P(x_0, y_0)$ , the formula $y - y_0 = m(x - x_0)$ can be used as procedure to determine the equation of the line that passes by $P$ . This mathematical connection is evident when rules, algorithms or formulas are used to arrive at a result (García-García & Dolores-Flores, 2021a).
5) Meaning (M)
This connection is presented when students associate a meaning to a mathematical concept, the interpretation that they have developed for these concepts, that is, what it means for them and what it represents, distinguishes it from another (García-García, 2019). In this sense, students express what the mathematical concept means to them, including their context of use or their definitions (García-García, 2019). In this research, we assume that this type can be more general, that is, we accept the existence of mathematical connection between meanings. We consider that this type emerges when the students relate different meanings attributed to a concept to solve a specific problem.
6) Reversibility (R)
It is present when a subject starts from a concept A to get to a concept B and invert the process starting from B to return to A (García-García & Dolores-Flores, 2021a). For example, this connection is established when the bidirectional relationship between derivative and integral, as operators, is recognized and when the Fundamental Theorem of Calculus is used as a way to link both concepts (García-García & Dolores-Flores, 2018).
7) Feature (F)
It is identified when the student manifests some characteristics of the concepts or describes their properties in terms of other concepts that makes them different or similar to others (Eli et al., 2011; García-García & Dolores-Flores, 2021a). For example, the derivative of a polynomial function is a reduction of its degree by one (García-García & Dolores-Flores, 2018).
8) Metaphorical (MT)
These connections are understood as the projection of the properties, characteristics, etc., of a known domain to structure another less known domain. For example, when the teacher or the student uses verbal expressions such as "travel through the graph without lifting the pencil from the paper" that implicitly suggest the conceptual metaphor "the graph is a path" (Rodríguez-Nieto et al., 2020).

example, giving a meaning of the derivative, representing it graphically or symbolically, and if possible giving examples. The subject reaches quality level 1 when the subject uses a formula to derive a function but does not say why he uses it and proceeds mechanically. Finally, the subject is at level 0 of quality of connections when, for example, he says that the derivative is the tangent line to the curve without emphasizing the slope of said line, or else, making errors in the procedure (Rodríguez-Nieto et al., 2021).

## METHODOLOGY

This is a qualitative research based on a case study (Padgett, 2016), which was developed in four phases: 1) selection of participants, 2) design of the questionnaire, 3) application of the questionnaire and task-based interview, 4) thematic analysis on quality of the intra-mathematical connections.

## Participants

Three students participated (21 - 23 years old), who were in the seventh semester of a mathematics career (university) at a higher-level school located in the capital of the state of Guerrero, Mexico. The selected students had taken and passed the Differential Calculus subject, especially the derivative topic.

## Instrument Design and Data Collection

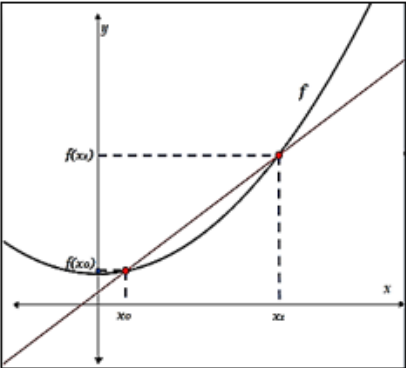
To collect data, we use the task-based interview, which allows focusing research on the development of mathematical tasks by the subjects, delving into the variety of important answers (topics) and knowledge associated with the learning process of the mathematics (Goldin, 2000). This method is useful because it allows us to evaluate the conceptual knowledge, motivate and consider the written productions and the justifications (arguments) of the students when they are solving the proposed tasks (Assad, 2015). A protocol with eight tasks was designed (see Figure 1). The first three explore

**Task 1.** Explain in your own words what is the derivative? |

**Task 2.** Do you consider that the derivative of a function  $f$  at an abscissa point  $x = x_0$  results from finding the slope of the tangent line to the graph of the function  $f$  at the abscissa point  $x_0$ ? Justify your answer.

**Task 3.** If the limit of a function in a point does not exist, what can the derivative of the function at the same point be said? Justify your answer.

**Task 4.** Use the following graphical representation to explain the definition of the derivative in a point.

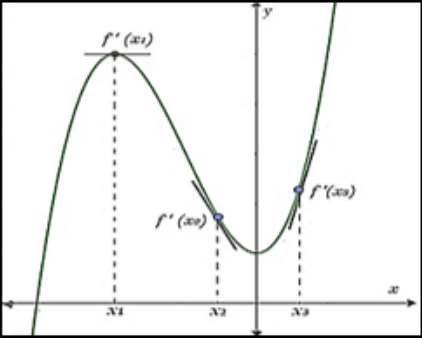


**Task 5.** Explain fully, why the derivative of a constant function is zero.

**Task 6.** Given the function  $f(x) = \frac{x^2}{4} + x + 5$ ,

- Find the equation of the tangent line to the graph of the function at  $x = 4$ .
- Determine the point of tangency. (Adopted from Rodríguez-Nieto et al. (2021)).

**Task 7.** From the following graphical representation, explain fully, how is the derivative of  $f$  in each case  $f'(x_1)$ ,  $f'(x_2)$  and  $f'(x_3)$ ?



**Task 8.** Given the function  $f(x) = x^3 + 8x^2 + 16x + 7$ ,

- Find the intervals at which  $f$  is increasing or decreasing.
- Determine where the function has relative maximum or minimum.
- Find the inflection points of the function.
- Determine where the graph of  $f$  is concave up or concave down.

Figure 1. Tasks protocol for data collection

the meanings of the derivative, through which we were able to identify the mathematical connections of type: between meanings, between different representations, part-whole and implication. In the fourth task, the geometric definition of the derivative is emphasized, to achieve procedural mathematical connections, between alternate representations, and implication mathematical connection type. The remaining tasks revolve around the application of the concept, for example, finding

maximums, minimums and concavity, in these tasks you can demonstrate procedural connections, between meanings, equivalent representations. The tasks were validated by users and by an expert. By users, a pilot was carried out with university students in mathematics. By experts, a professor specialized in Calculus and Mathematical Analysis, analyzed the tasks according to the objective of the research.

**Table 2.** Frequency of the intra-mathematical connections made by S1, S2 and S3 Types of intra-mathematical connections

Tasks		T1	T2	T3	T4	T5	T6	T7	T8	fS
DR	S1	1	1	1	1	1	0	0	2	7
	S2	1	1	1	1	1	0	3	2	11
	S3	2	1	2	1	1	1	3	3	14
PW	S1	0	0	1	0	1	0	0	0	2
	S2	1	1	1	0	1	0	0	0	4
	S3	2	0	1	0	1	1	0	0	5
I	S1	0	0	1	1	1	0	1	1	5
	S2	0	0	1	0	1	1	3	1	7
	S3	1	0	1	0	1	1	3	2	9
P	S1	1	1	1	0	1	0	0	2	6
	S2	1	1	1	0	1	2	3	4	13
	S3	2	1	2	0	1	3	3	5	17
M	S1	1	1	1	1	1	0	0	0	5
	S2	1	0	1	1	1	1	1	1	7
	S3	2	1	1	1	1	1	1	0	8
R	S1	0	0	0	0	0	0	0	0	0
	S2	0	0	0	0	0	0	0	0	0
	S3	0	0	0	0	0	0	0	0	0
F	S1	0	0	3	0	0	0	0	0	0
	S2	0	1	2	0	0	0	0	0	0
	S3	0	0	2	0	0	0	0	0	0
MT	S1	0	0	1	2	0	0	0	0	0
	S2	0	0	1	0	1	0	1	0	0
	S3	0	0	1	0	0	0	0	0	0
fT	S1	3	3	9	5	5	0	1	5	31
	S2	4	4	8	2	6	4	10	8	46
	S3	9	3	10	2	5	7	10	10	56

Note. In Table 2, *fT* means the frequency of the intra-mathematical connections made in each task and *fS* is the frequency of the intra-mathematical connections made by students for each type of connections

In the application of the instrument the researcher and the student interacted through the proposed tasks. A questionnaire where the eight tasks were raised was provided to each case study. While the case study (interviewee) solved each task, the researcher (interviewer) delved into the student's reasoning, posing questions related to the task he was solving.

### Data Analysis

The analysis was carried out in six phases of thematic analysis (Braun & Clarke, 2006). In the first phase (*familiarization with the data*), the written productions of the students and the video recordings of the interviews were considered, which were transcribed with the purpose of familiarizing us with the data. In the second phase (*generating initial codes*), we looked for propositions to infer mathematical connections in the form of codes, words and phrases, for example, a mathematical connection of meaning is evidenced in the following interview excerpt: "the derivative is the slope of the tangent line to the curve in a point". In the third phase (*searching for themes*) we group the codes established in the previous phase according to the type of mathematical connections. In the fourth phase (*reviewing themes*) the themes were reviewed identifying that the grouped codes correspond exactly to a theme or intra-mathematical connection. Also, we considered the field

notes, videos, and written productions for the triangulation of the data taken from different sources of information collection and triangulation among researchers to increase the quality and validity of the data, given that different perspectives of analysis are presented, and the bias of a single researcher is eliminated (Aguilar & Barroso, 2015). The fifth phase (*defining and naming themes*) referred to the definitive naming of the themes, that is, that they were related exactly to a mathematical connection and its quality. Finally, in the sixth phase a *report* is made of the findings, which in this case is the quality of the intra-mathematical connections (see the *Results* section).

## RESULTS

This section emphasized the intra-mathematical connections' quality made in the resolution of the tasks (Figure 1) for each case study. From now on, students will be named as follows: case 1 (S1), case 2 (S2), case 3 (S3) and the interviewer (I). Table 2 shows the frequency with which each case study made each mathematical connection in the resolution of the tasks.

### Mathematical Connections made by S1

To solve task 1, S1 made mathematical connections between two meanings, such as the slope of the tangent

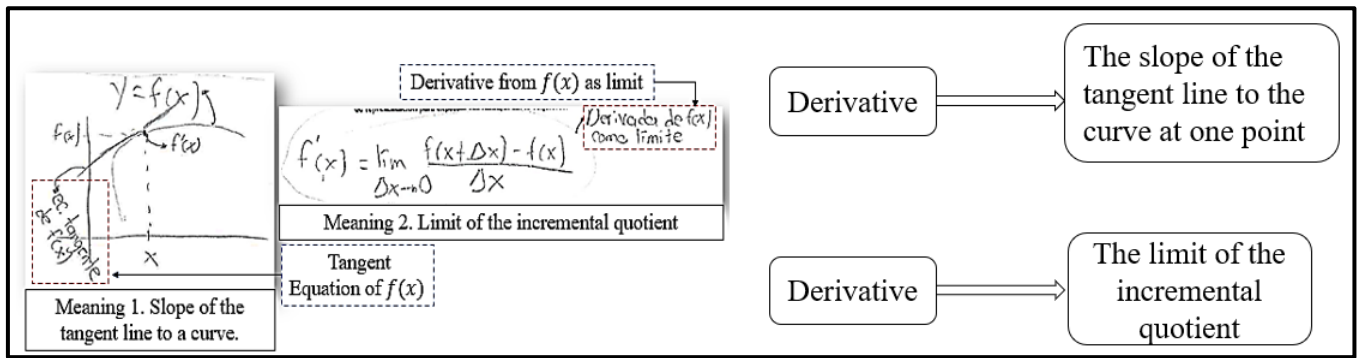


Figure 2. Mathematical connections of different representations and meaning

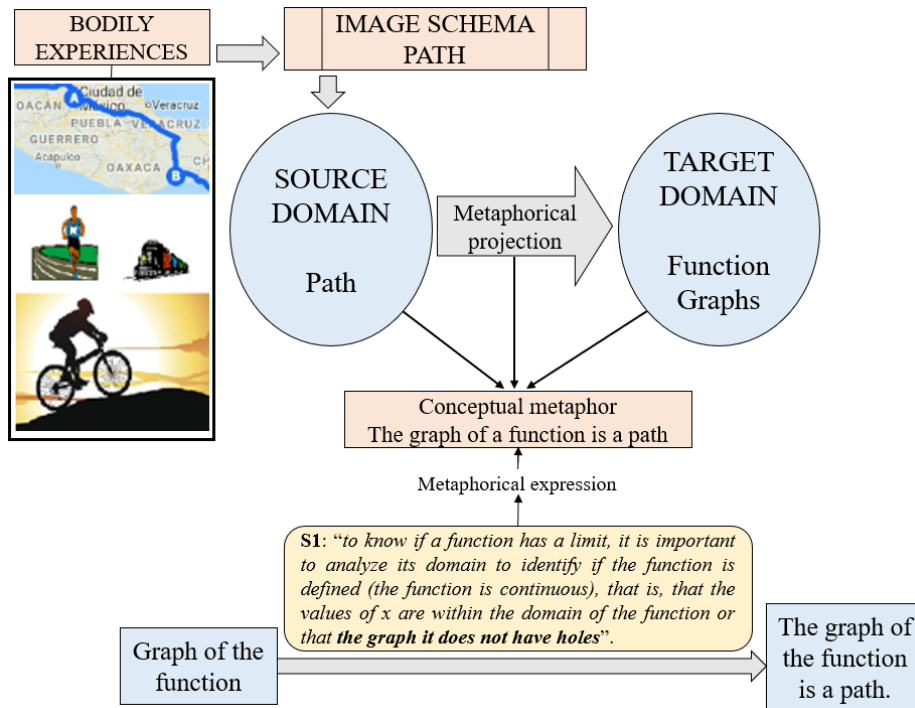


Figure 3. Metaphorical connection scheme (Adopted from Rodríguez-Nieto et al., 2020)

line to the curve in a point (meaning 1) and as the limit of the incremental quotient (meaning 2). In addition, connections of different representations type were evidenced (graphical and algebraic), because S1 placed the notations of the function and the derivative and related them to their respective graph referring to meaning 1, and related the graph and meaning 1 with a different register of meaning 2 (see Figure 2).

For the arguments put forward by S1 (read excerpt from the interview), the quality of the mathematical connections that he established are located at a level 2.

I: What is the derivative?

S1: For me the derivative [is a] limit, that is, you have a function  $f(x)$  and if you want to calculate its derivative this can be done [obtaining] the limit of a  $\Delta x$  when it tends to zero of [the difference of the variation of]  $f(x)$  plus  $\Delta x$  minus  $f(x)$  over  $\Delta x$ . [But, in addition] the derivative, what it

represents, is the slope of the tangent line at one point.

In task 2 it was shown that S1 made the mathematical connection between two meanings of the derivative as the slope of the tangent line and the instantaneous rate of change (meaning 3). Likewise, mathematical connections of procedure were identified when relating the right triangle to the slope to explain the derivative and the connection of different representations by showing the relationship between the graph of the derivative (meaning 1) and a right triangle as a form of studying the slope. Both mathematical connections made by S1 are located at a level 2 in quality. In task 3, a mathematical connection of feature was evidenced when S1 stated that "to know if a function has a limit, it is important to analyze its domain to identify if the function is defined (the function is continuous), that is, that the values of  $x$  are within the domain of the function or that the graph it does not have holes", referring to the metaphorical connection (see Figure 3).

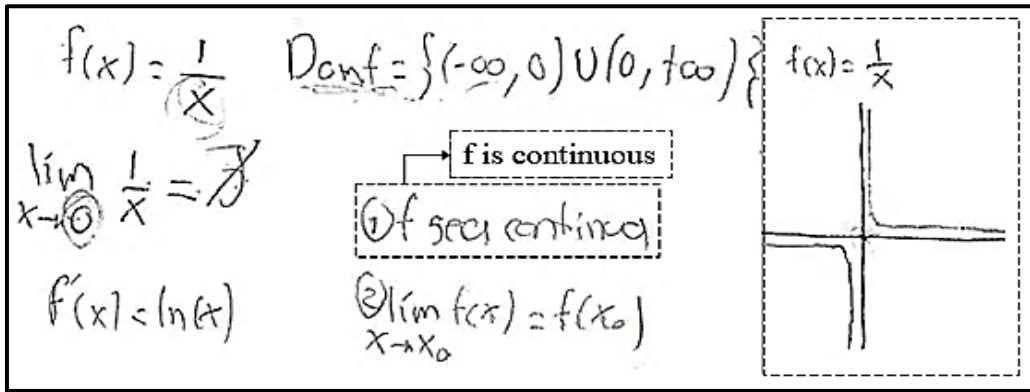


Figure 4. Mathematical connections between different representations and part-whole

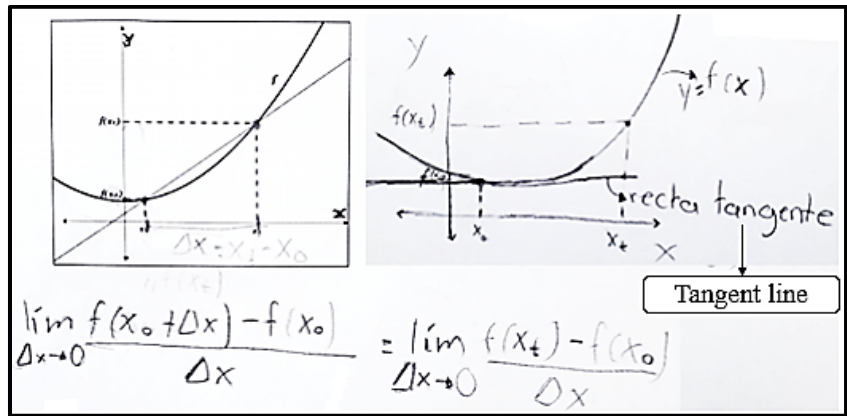


Figure 5. Derivative as the limit of the incremental quotient and its connection with the graph

S1 then confirms that there is a relationship between continuity and the existence of the limit, mentioning that there are two criteria for the limit to exist: that the function is continuous and that the value of the limit when  $x \rightarrow x_0$  of  $f(x)$  is equal to the function evaluated at that point  $x_0$ . In this sense, the mathematical connection of implication type is evident when S1 mentioned that the limit exists at one point, if the function is continuous at that same point. And if the function is not continuous at one point, at that point there is no limit and, as a consequence there is no derivative, which evidences a mathematical connection of feature type. Likewise, S1 exemplified with the undetermined limit when  $x \rightarrow 0$  using the function  $f(x) = \frac{1}{x}$  (mathematical connection of feature), observing the use of connections of different representations, part-whole, implication, meaning, feature and metaphorical type, with a quality level 2 (see Figure 4).

In task 4, S1 established three mathematical connections: between the meanings of 1 and 2 addressed in the previous tasks, that of different representations (the graph provided in task 4 with the limit of the incremental quotient corresponding to meaning 2 and the slope of the tangent line to the curve corresponding to meaning 1), and the type of procedural connection when using the graph as a means to establish the connection between two meanings. Likewise, in the

verbal arguments of S1, the process of transition from the secant line to the tangent line that is outlined in a graph (connection of different representations), was identified. However, in the graph located in Figure 5 the approximations of the line are not shown when the moving point approaches the fixed point until the line becomes tangent to the curve, but S1 justified his response verbally (read extract of interview). The mathematical connections made by S1 are located at a quality level 2.

S1: Depending on how this value limit is, secant line will tell us how much it will be inclined in such a way that it is no longer a secant line, but a tangent line, and when it becomes inclined, we can see that this tangent line is going to touch at point  $x_0$  and the limit will be the slope of the tangent equation.

I: Does the secant line become tangent?

S1: We have a limit,  $\Delta x$  is approaching zero, that as we take values very close to zero on the right or on the left is how that secant line is transformed into a tangent line, we are approaching zero on the right and on the left and the limit value it tells us that both this secant line is going to tilt so that it is already transformed into a tangent line (Mathematical connection of implication type).



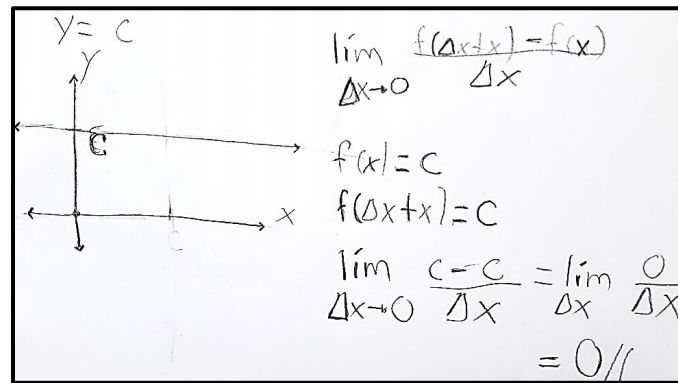


Figure 6. Derivative of a constant function and procedural connection

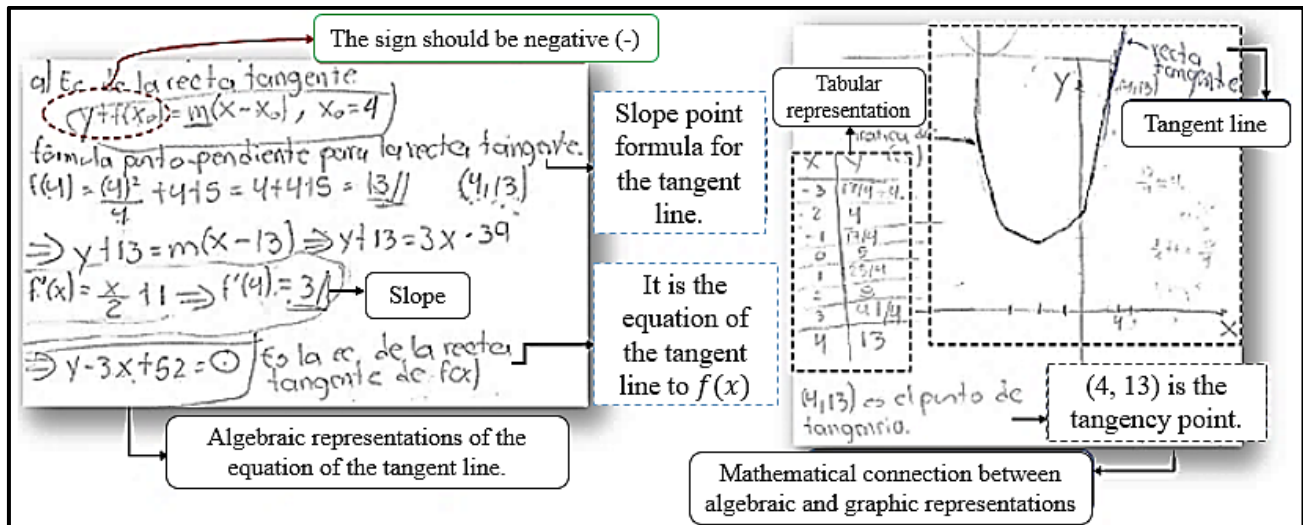


Figure 7. Resolution of S1 to find the equation of the tangent line

Furthermore, when S1 referred to “(...) approaching zero on the right and on the left and the limit value (...)”, made a metaphorical connection of a fossilized type as presented in Rodríguez-Nieto et al. (2020).

In task 5, S1 made a procedural connection since it uses the derivative and the four-step rule to find the answer. Also, the connection of different representations type was observed (graph-algebraic), see Figure 6.

S1 in its verbal arguments (read interview extract) manages to justify the graphical-algebraic process carried out, the mathematical connections of implication type are also evident when S1 stated that “if the limit is equal to zero the derivative will be equal to zero and for every constant function the derivative is zero”. On the other hand, S1 used the mathematical connection of part-whole type when specifying that  $y = c$  is a constant function whose derivative is equal to zero and made a demonstration using the definition of limit and generalizing that for all constant function, the derivative is equal to zero. The mathematical connections made by S1 in task 5 were placed at a level 2 of connection quality.

S1: In already more algebraic terms, if we consider the definition of derivative by limit, this from here  $\Delta x$  tends to zero  $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$  if  $f(x + \Delta x) = c$  y

$f(x) = c$ , then the limit will be equal to zero and the derivative will always be zero.

To solve task 6, S1 used the power formula to find the derivative at a point and thus obtain the slope (mathematical connection of procedure). Likewise, the mathematical connection of implication was identified when S1 mentioned that “if the definition of the derivative allows us to find the slope of the tangent line, then it is a way of how to find the slope to substitute in the equation the slope point to find the tangent line”. S1 revealed other procedural mathematical connections when he used the algebraic expression of the function by replacing the value of  $x_0$  to obtain  $y_0$  and find the point of tangency, and when he used the equation slope point to find the slope of the line. In addition, S1 used different representations when graphing the function that in turn related to the derivative as the slope of the tangent line to the curve in a point. However, S1 in his procedure presented an inconsistency in the slope point formula ( $y + f(x_0) = m(x - x_0)$ ) being basic knowledge to find the answer, leading to obtaining a wrong answer (see Figure 7). Consistent mathematical connections were evident, however S1 made a connection that was located at level 0 of quality.

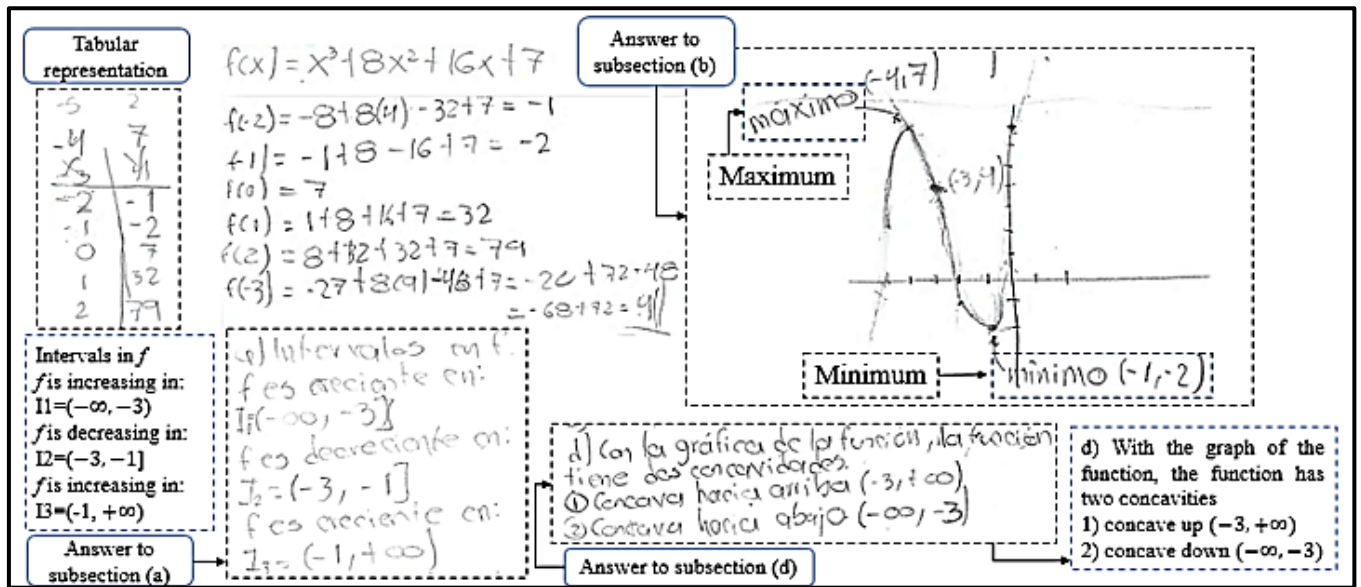


Figure 8. Evidence of resolution to task 8

Table 3. Quality level of mathematical connections made by S1

Type of connection	Tasks							
	T1	T2	T3	T4	T5	T6	T7	T8
DR	L2	L2	L2	L2	L2	L0		L1
PW			L2		L2			
I			L2	L2	L2		L0	L1
P		L2	L2	L1	L2	L0	L0	L1
M	L2	L2	L2	L2	L2			
R								
F			L2					
MT			L2	L2				

Note: Different representations (DR); Part-whole (PW); Implication (I); Procedural (P); Meanings (M); Reversibility (R); Feature (F); Metaphorical (MT). Level 0 (L0), Level 1 (L1) and Level 2 (L2)

To answer task 7, S1 used the graphical representation to analyze each case (mathematical connection of procedure). S1 explained that in the first case the slope of the tangent line is positive, being an incorrect answer since at that point the slope is equal to zero. For case 2, another inconsistency was noted to mention that the function is concave down. However, S1 established procedural connections by relating the graph with the derivative (meaning 1) assuming that, if the slope is negative, the derivative is also negative (mathematical connection of implication). In the third case S1 ensures that the slope is positive because it grows as in the first case. A disconnection was observed in the responses of S1, reaching a level 0 of quality.

S1 responded to task 8 by showing that the function is increasing in the intervals  $(-\infty, -3)$  and  $(-1, +\infty)$  and decreasing in the interval  $(-3, -1]$ , evidencing the connection between different representations. Mathematical connections of procedure were evidenced when S1 used the algebraic expression and a table of values to graph the function. Then, S1 mentions that the maximum point is  $(-4, 7)$  and the minimum is  $(-1, -2)$ . Also, he mentioned that the graph has two concavities: upwards in  $(-3, +\infty)$  and downwards  $(-\infty, -3)$ ,

observing an approximate answer given that S1 used the graph to refer to the inflection point (mathematical connection of procedure). It should be noted that the inflection point is at the abscissa point  $x = -\frac{8}{3}$  and is essential to properly determine the concavity (see Figure 8).

In this case, S1 did not use the derivative to solve task 8 and said he did not remember what an inflection point was to respond to subsection (c). In this sense, the mathematical connections made by S1 to answer task 8 are located at a quality level 1. Next, Table 3 summarizes the level of quality of the mathematical connections (L) established by S1 when solving the 8 mathematical tasks proposed.

### Mathematical Connections made by S2

In the resolution of task 1 it was evident that S2 conceived the derivative as the slope of the tangent line to the curve in a point (meaning 1) and as the limit of the incremental quotient (meaning 2), in this process, connections were observed between meanings of the derivative, of different representations type where it relates the graph of the derivative with the algebraic

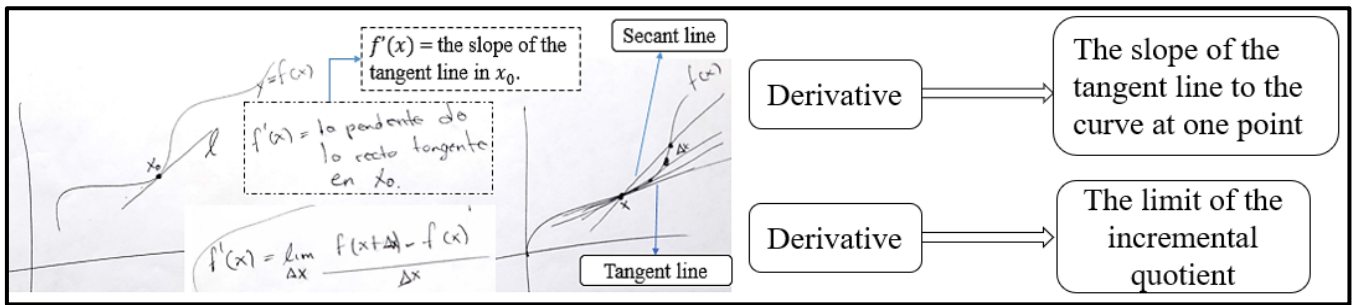


Figure 9. Mathematical connections between different representations and between meanings

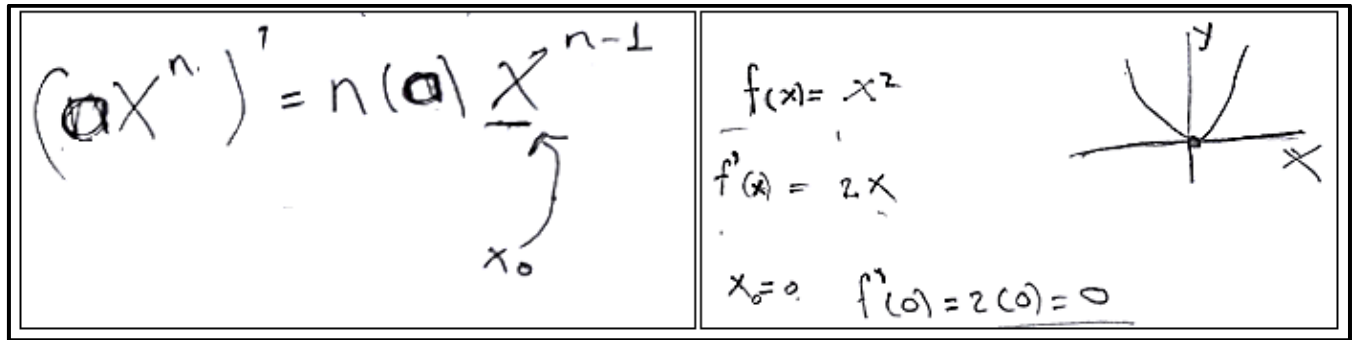


Figure 10. Use of the part-whole and different representations mathematical connections

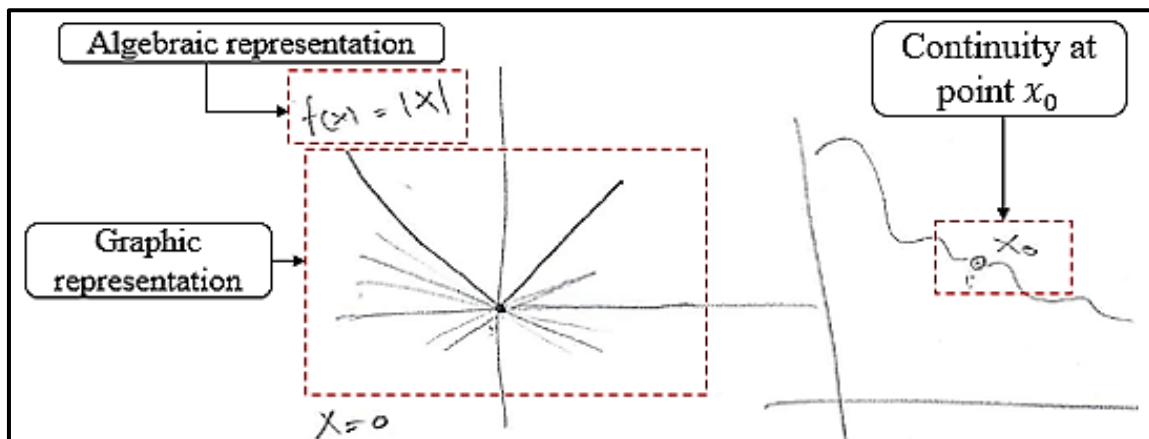


Figure 11. Evidence of mathematical connections of different representations type

representations  $\lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$  and of procedural connection because S2 made a graph to represent the transit from the secant line to the tangent line to the curve associated to the function and mentioned that the derivative is obtained by means of the limit of the incremental quotient (see Figure 9). The mathematical connections made in the resolution of task 1 are located at a quality level 2.

In response to task 2, S2 mentioned that the derivative can be obtained by means of the power formula  $\left(\frac{dx^n}{dx} = nx^{n-1}\right)$  to obtain derivatives of polynomial functions, which is a reduction of its degree by one, showing a mathematical connection of feature type. In that sense, S2 used the part-whole of mathematical connection type because he recognized that the above formula is a general case and the particular case is presented when he uses it to derive the

function  $f(x) = x^2$ , finding that  $f'(x) = 2x$ . Then, by means of the mathematical connection of procedure, he evaluated  $x_0 = 0$  in the algebraic expression of  $f'$  ensuring that  $f'(0) = 0$  is the slope of the tangent line to the graph at  $x = 0$  (see Figure 10). The mathematical connections made by S2 were placed at a quality level 2.

In response to task 3, S2 used the mathematical connection of implication type when he explained that "if the derivative at one point does not exist, then the function is not continuous". Also, mathematical connections of feature and part-whole type were identified (see Figure 11) when S2 exemplified with the absolute value function, specifying that at the abscissa point  $x_0 = 0$  the derivative does not exist because the limit does not exist, and the lateral limits are different, and the absolute value function is not derivable at a point  $x = 0$ . Furthermore, S2 established the connection of different representations when you related a  $f(x) = |x|$  with the

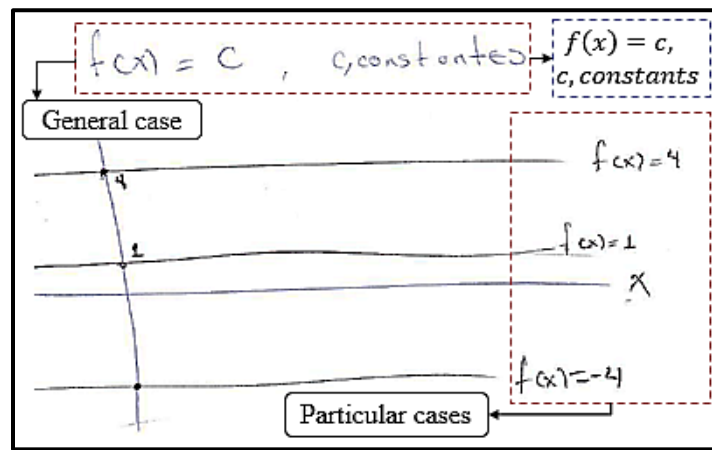


Figure 12. Mathematical connections of part-whole, particular cases of constant function

graph of the absolute value function and the graphical representation of continuity at one point (read interview extract). In addition, S2 made a fossilized-type metaphorical connection when he argued that “if we calculate the limit on the left or on the right and they are different, so you can say that there is no limit”.

S2: If the limit does not exist, we can say that the derivative does not make sense at that point. If we calculate the limit on the left or on the right, we realize that if we approach here it gives us one, and if we approach this one here, it gives us another.

I: How much does it give you?

S2: One and minus one, and has different limits, therefore, the limit does not exist at that point and from that we conclude that the derivative does not make sense at that point ... in some part, that the function is not continuous.

Subsequently, S2 made mathematical connections of meaning type when he mentioned that the derivative is the slope of the tangent line and established the mathematical connection of feature type when he said that infinite lines can pass through the point with abscissa  $x = 0$  and for that reason there is not the derivative at that point (see the following interview extract).

I: What do you mean by continuous?

S2: (...) a function is continuous if it has a derivative in all its points, a clear example would be  $f(x) = |x|$  that at  $x = 0$  its derivative does not make sense because the derivative is the slope of the tangent line, but if we look at this, it can be a tangent and this other one can be a tangent, this one can finally have an infinite number of tangents, which contradicts, as the derivative is the slope of the tangent line at this point and here we are finding that it has many tangents.

The mathematical connections made by S2 in the solution of task 3, were placed at a level 2 of quality.

In task 4, a mathematical connection of different representations type (graphical-algebraic and graphical-graphical) was found because S2 uses the graph to explain the derivative considering the meaning of the limit of the incremental quotient (mathematical connection of meaning type), placing secant lines that are approaching a tangent line at an abscissa point  $x_0$ . In this sense, when the mobile point of the secant lines coincides with the  $x_0$ , it is the derivative. These types of connections are located at a quality level 2. In the resolution of task 5, S2 uses the algebraic representation  $f(x) = c$  as a general case of a constant function and exemplifies with particular cases about its conception of the derivative of a constant function, evidencing the mathematical connection of part-whole and between different representations type (see Figure 12). Finally, by means of the mathematical connection of implication type, S2 recognized that the derivative will always be zero for any constant function where the line will always be horizontal or parallel to the  $x$ -axis, that is, S2 made a metaphorical mathematical connection of an orientation type (not fossilized) when it referred to the horizontal or parallel line as evidenced in Rodríguez-Nieto et al. (2020). In this sense, the mathematical connections addressed in this task are at a quality level 2.

For the resolution of task 6, S2 established mathematical connections of procedure. He determined the slope of the line using the derivative of  $f$  (mathematical connection of meaning type) and mathematical connection of implication since to obtain the equation of the tangent line implies the use of the derivative and the slope point equation. In addition, he found the point of tangency by evaluating the value of  $x = 4$  in the algebraic representation of  $f$ , and then, S2 found the equation of the line, substituting the point of tangency  $(4, 13)$  and the slope  $m = 3$  in slope point equation  $y - y_0 = m(x - x_0)$ , to obtain the equation  $y = 3x + 1$  (mathematical connection of procedure). In this

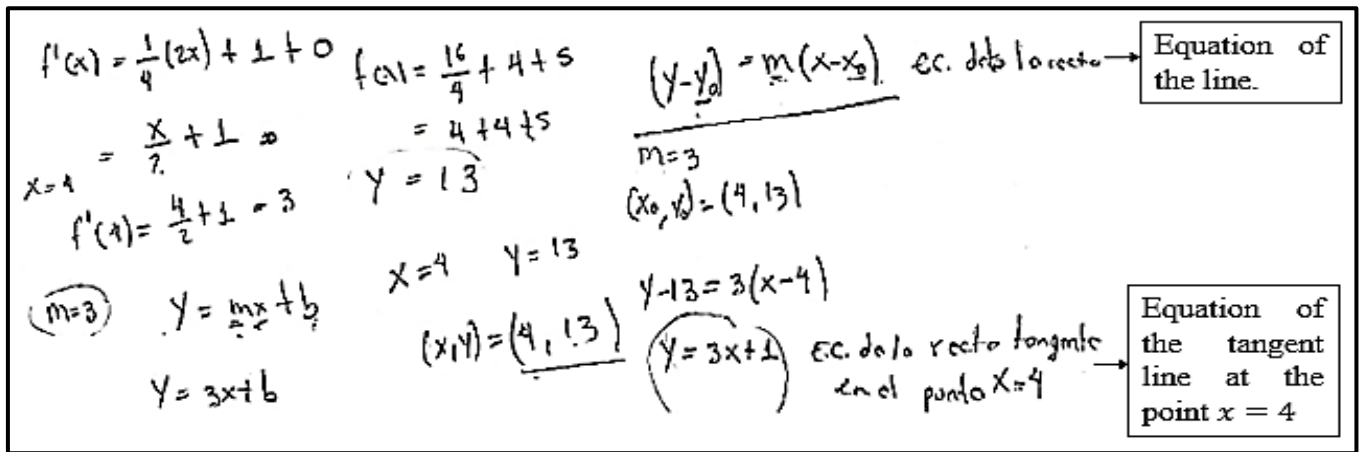


Figure 13. Mathematical connections of meaning, procedural and implication type

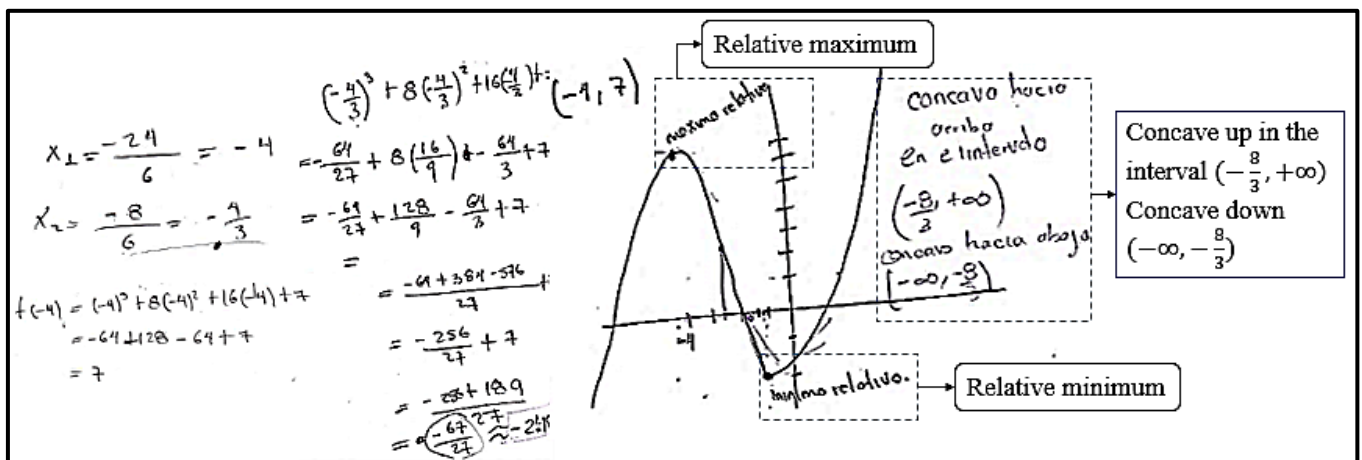


Figure 14. Connections of procedural and different representations type

case, the mathematical connections of S2 were placed at a quality level 2 (see Figure 13).

In response to task 7, mathematical connections of procedure type (algebraic processes to relate the slope of the line to the derivative) and of different representations were evidenced because S2 used the graph to consider that in  $f'(x_1)$  the slope of the tangent line is zero because it is a line parallel to the  $x$ -axis that there is a possibility of a maximum or a minimum of the function, identifying mathematical connections of meaning and metaphorical type and the mathematical connection of implication can be seen when S2 argued that "as  $f'(x_2)$  is negative then the function is decreasing in  $x_2$ ", but he also argued that "as  $f'(x_3)$  is positive then the function is increasing in  $x_3$ ". In this sense, the mathematical connections established by S2 were placed at level 2 of quality.

To solve task 8, S2 made mathematical connections of procedure, since it derived the function  $f$  to find where the derivative becomes zero, in addition, the relationship between the derivative function and the general formula for solving quadratic equations was identified, to find the values of the abscissa where  $f$  will have maximums or minimums. Once the zeros were obtained, he sketched the graph of the function, starting by

evaluating  $x_1 = -4$  and  $x_2 = -\frac{4}{3}$  in the algebraic representation of the function  $f(x) = x^3 + 8x^2 + 16x + 7$  relating it to the graphical representation of  $f$  where the mathematical connection of different representations type was evidenced (see Figure 14). It was identified that the mathematical connections of procedure established so far, were located at a quality level 2.

Subsequently, to find the inflection point, S2 did not go to the second derivative, but instead used the distance formula (mathematical connection of different representations (graphical-algebraic) and procedure) considering points A  $(-4, 0)$  and B  $(-\frac{4}{3}, 0)$ , obtaining  $\frac{8}{3}$  equal to the distance between A and B. Then, establishing the mathematical connection of procedure found the midpoint equivalent to  $\frac{4}{3}$  and finally added  $\frac{4}{3}$  a  $-4$ , finding that the inflection point has the coordinates  $(-\frac{8}{3}, f(-\frac{8}{3}))$ , evaluating  $x = -\frac{8}{3}$  in  $f$ , evidencing connections between different representations (graphical-algebraic). It should be noted that S2 uses a recursive process that is not commonly used to find the inflection point. Therefore, the mathematical connections of procedure type established by S2 are at a

Table 4. Quality level of mathematical connections made by S2

Type of connection	Tasks							
	T1	T2	T3	T4	T5	T6	T7	T8
DR	L2	L2	L2	L2	L2		L2	L2
PW		L2	L2		L2			
I			L2		L2	L2	L2	L2
P	L2	L2	L2		L2	L2	L2	L1
M	L2	L2	L2	L2	L2	L2	L2	L2
R								
F		L2	L2					
MT			L2		L2		L2	

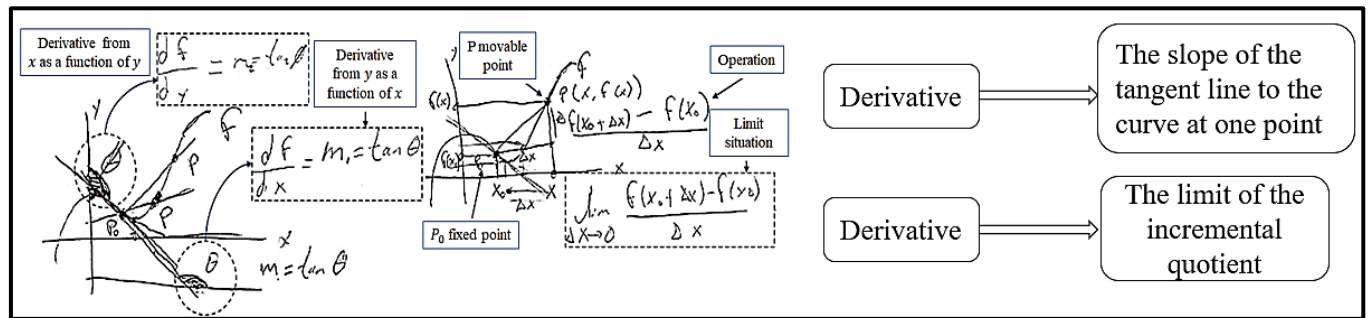


Figure 15. Connections between different representations of the derivative

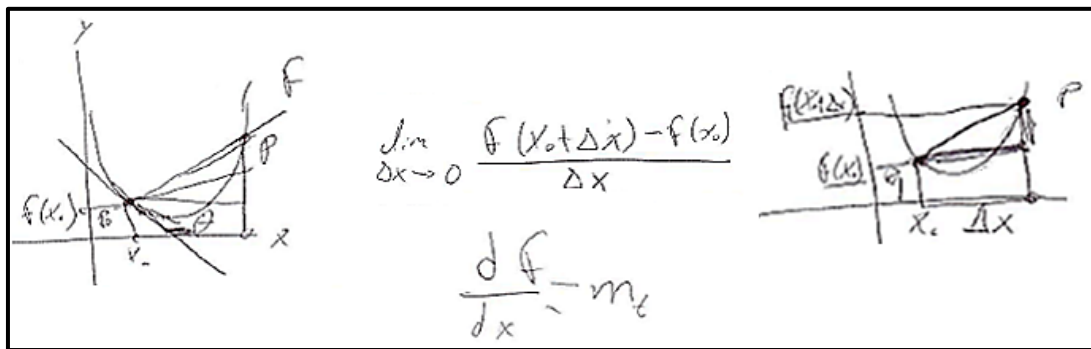


Figure 16. Mathematical connections between different representations

quality level 1. Table 4 summarizes the quality levels of the mathematical connections (L) made by S2.

**Mathematical Connections made by S3**

In the resolution of task 1, S3 established mathematical connections between meanings associated with the derivative such as the limit of the incremental quotient (meaning 2) and the slope of the tangent line to the curve (meaning 1). It also interprets the slope as the trigonometric angle tangent (meaning 4), see Figure 15. The connection between different representations (graphical-algebraic) related to each meaning was identified. It was shown that S3 highlights two cases: the first one, when the derivative is with respect to x with the angle  $\theta$ , and the second case is the derivative with respect to y with the angle  $\phi$  (see Figure 15). In Figure 15, it is observed when S3 revealed key aspects of the meaning of the derivative: the first aspect refers to the quotient of increments  $\left(\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}\right)$  (S3 called this quotient an 'operation'); the second aspect that he

mentioned was that, if the limit is applied to the increment quotient, then it is considered a limit situation (mathematical connection of implication type), and third, S3 indicated that, in the process of approximating the derivative at a point, there is a 'movable point  $p_0$ ' that is approaching another fixed point  $p$ , evidencing the mathematical connection of different representations. The mathematical connections established by S3 were placed at level 2 quality.

In the case of task 2, S3 considered that the derivative is the slope of the tangent line to the curve of a function at a point. S3 also mentioned that point  $p$  overlaps  $p_0$ , but then said that point  $p$  cannot be superimposed because after the limit is undetermined and the distance between  $x$  and  $x_0$  is given by  $|x - x_0| < \epsilon$ , where  $\epsilon > 0$ . In addition, it is noted that S3 made mathematical connections between meanings and different representations type when, by means of the graph, he explains the meanings and the transition from the secant to the tangent line (see Figure 16). The connections evidenced in task 1 were placed at a quality level 2.

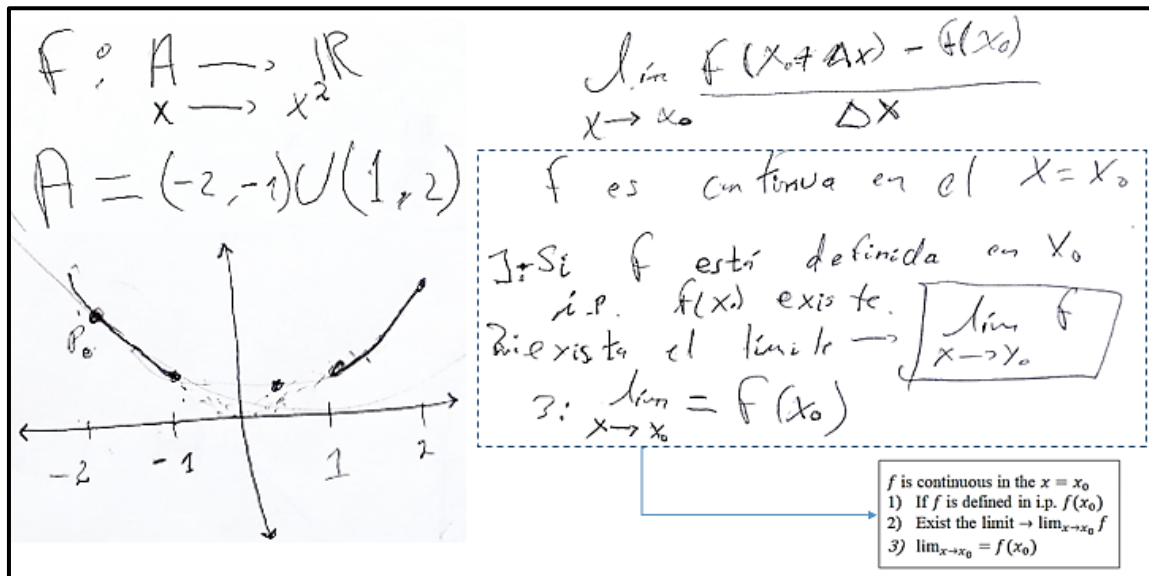


Figure 17. Connections of implication, different representations, and part-whole type

In relation to task 3, S3 made mathematical connections of feature type, arguing that the derivative does not exist if the limit on the right is different from the limit on the left or that the function is not defined or indeterminate at a point  $x_0$  where it is desired to obtain the derivative. Likewise, the metaphorical connection of the fossilized type was evidenced when S3 referred to the limit on the right and on the left. Next, establishing the mathematical connections of procedure and different representations type, S3 exemplified in the same way as S1, stating that the function  $f(x) = \frac{1}{x}$  for  $x = 0$  is not defined, so its derivative does not exist, evidencing also mathematical connections of feature, implication and part-whole type, since it is a particular case of a rational function. Likewise, the procedural and part-whole connections were evidenced in the example of a function by parts indicating that the derivative exists when the function is continuous at a point, and if the function is discontinuous, is not derivable (Mathematical connections of implication or logical relationships), see Figure 17. The connections evidenced in the response to task 3 are at quality level 2.

In the case of task 4, the mathematical connection of different representations type was observed when S3 used the graph to specify that there is a fixed point and a mobile point that changes value, so that  $0 < |x_0 - x_1| < \varepsilon$  for all  $\varepsilon > 0$  (mathematical connection of meaning type), that is, that the distance is reduced being greater than zero and less than a positive number. Consequently, the lines get closer until they approach the tangent line to the curve. In this process, mathematical connections of part-whole type are also evidenced in each case of secant lines. These types of mathematical connections were placed at a level 2 quality.

About task 5, S3 made mathematical connections of procedure (procedure to find the limit) and connections

of meaning type, he used the meaning of the derivative as the limit of the incremental quotient to demonstrate why the derivative of a constant function is equal to zero. Furthermore, mathematical connections between different representations (graphical-algebraic) were evidenced. Likewise, part-whole connection was evidenced when in the demonstration carried out, S3 emphasized in a general case  $f(x) = k$  and mentioned that there are particular cases and affirmed that, if the function is constant, then its derivative is zero  $\frac{df(x)}{dx} = 0$  (mathematical connection of implication type), see Figure 18. The connections established by S3 in task 5, were placed at a quality level 2.

S3 uses the meaning of the derivative as the slope of the tangent line to solve task 6. Likewise, he argued that the function  $f(x) = \frac{x^2}{4} + x + 5$  is a particular case of polynomial functions (connections of part-whole) and continuous in all its points. S3 also argued that "if the function is continuous, at one point it is derivable" evidencing the use of the mathematical connection of implication type (see interview extract).

S3: Since the function has no restrictions, then we are going to put as domain all the real numbers. Graphically it is seen that it is a parabola or in general it is a polynomial, and we already know that all polynomials are continuous, so it has a derivative at any point.

Then, through the mathematical connection of procedure S3 found the derivative to find slope  $m$ , where the mathematical connection of meaning was also evidenced (use of geometric meaning). Subsequently, he used the mathematical connection of procedure by replacing  $x = 4$  in the algebraic expression of  $f$  to obtain the point of contact or tangency  $P_0 = (4, 13)$ . S3 used the slope point formula to find the equation of the tangent

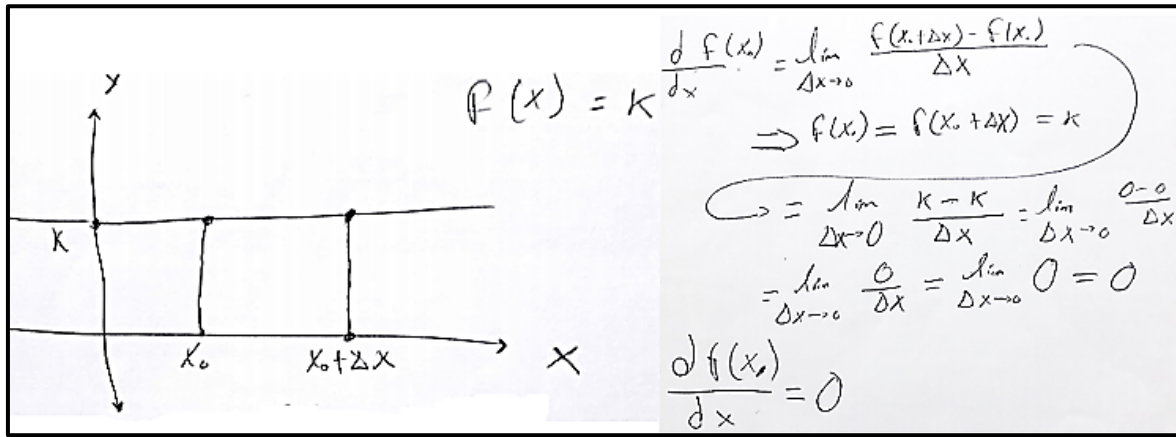


Figure 18. Demonstration of the derivative of a constant function (procedural connection)

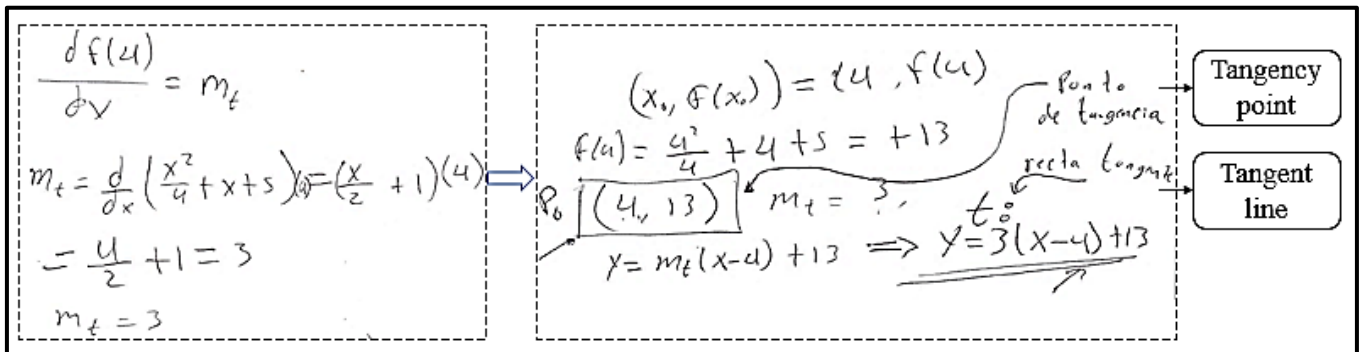


Figure 19. Procedural connection to find the equation of the tangent line

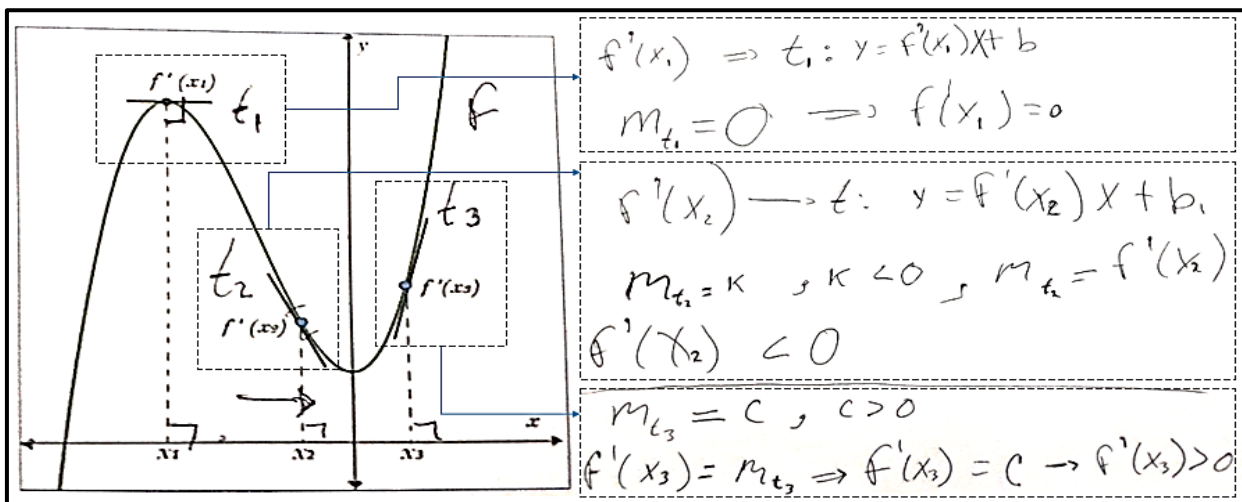


Figure 20. Mathematical connections of implication type

line to the curve of  $f$ , see Figure 19. The mathematical connections observed in the resolution of task 6 were located at level 2 of quality, since they are argued connections and consistent procedures.

In the resolution of task 7 (see Figure 20), S3 made mathematical connections of meaning type, since he used the meaning of the derivative as the slope of the tangent line to determine the behavior of the function at a point and the mathematical connection of implication when he related the sign of the slope and the derivative. In the first case, he considered that the derivative is equal

to zero and related it to the formula of the equation slope intercept point (mathematical connection of procedure). For the second case S3 mentioned that the derivative of the function is less than zero and for the third case it ensures that the derivative at the abscissa point  $x_3$  is greater than zero. Also, the mathematical connection of different representations type was observed to respond to the task using the graph. These connections of S3 were placed at level 2 of quality.

To solve task 8, using the mathematical connection of procedure type, S3 used the derivative to know where



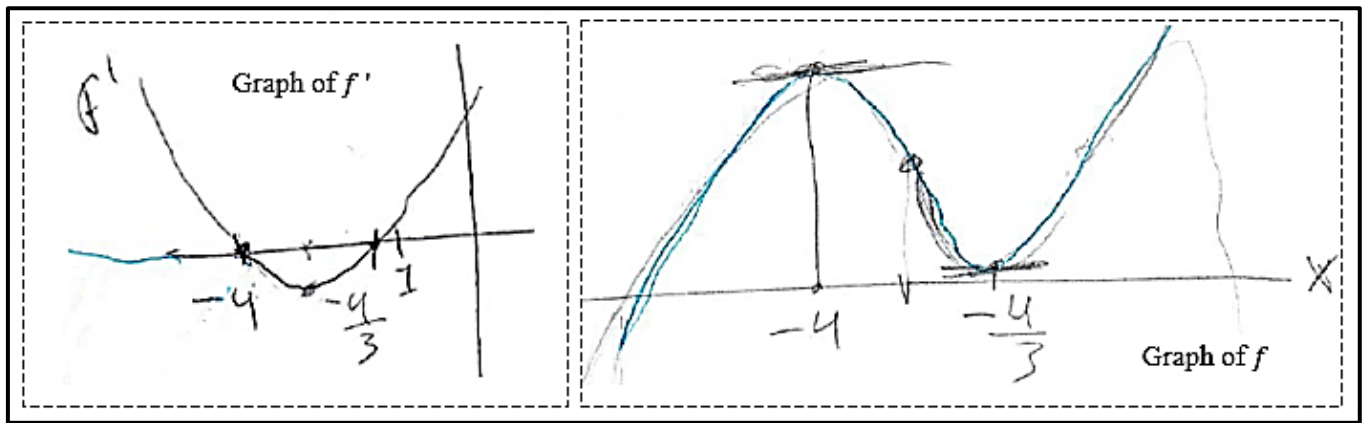


Figure 21. Connection of implication and different representations type (graph of  $f'$  and the  $f$ )

Table 5. Quality level of mathematical connections made by S3

Type of connection	Tasks							
	T1	T2	T3	T4	T5	T6	T7	T8
DR	L2	L2	L2	L2	L2	L2	L2	L2
PW			L2		L2	L2		
I	L2		L2		L2	L2	L2	L2
P	L2	L2	L2		L2	L2	L2	L1
M	L2	L2	L2	L2	L2	L2	L2	L2
R								
F			L2					
MT			L2					

the function has relative maximum or minimum. He made a mathematical connection between the quadratic equation obtained by deriving  $f$  and the rule of completing square allowing him to find the maximum points at  $x = -4$  and minimum at  $x = -\frac{4}{3}$  (connections of procedure type). S3 assured that the function is increasing at the intervals  $(-\infty, -4)$  y  $(-\frac{4}{3}, +\infty)$  and  $f$  is decreasing in the interval  $(-4, -\frac{4}{3})$ . So far, it was identified that the Mathematical connections of procedure are located at a quality level 2. Also, like S2, S3 used the midpoint formula to find the inflection point with abscissa  $x = \frac{8}{3}$ , evidencing the mathematical connection of procedure type. Once the inflection points were obtained, S3 pointed out that the function is concave down in  $(-\infty, -\frac{8}{3})$  and concave down in  $(-\frac{8}{3}, +\infty)$  (mathematical connections of different representations and procedural). However, S3 did not use the second derivative to find the inflection points, but instead went to a recursive process (midpoint formula) that was adequate for the behavior of the function. Therefore, the mathematical connections of involvement are located at level 1 quality. In addition, S3 in his reasoning not only emphasized algebraic representation, but made mathematical connections between different representations, considering the graphs of derivative  $f'$  and the original function  $f$  (mathematical connection of implication and procedural), allowing him to sketch the graph of  $f$  from the information offered by the graph of  $f'$  (see Figure 21).

It was evidenced that the mathematical connections made by S3 in the resolution were placed at level 2 of quality.

Table 5 shows the quality levels of mathematical connections (L) made by S3.

## DISCUSSION

The results indicated that the case studies reached a different level of connection quality according to the proposed task. In this sense, the level reached by each case study was associated with the use of properties, theorems, and knowledge of Calculus, in particular, on the derivative. Likewise, six types of connections were identified in the written productions and in the verbal arguments of S1, S2 and S3: procedural type, different representations, implication, metaphorical, meanings and part-whole. From these results, it was identified that S1, S2 and S3 made type connections between different representations (graphical-algebraic) and mathematical connections between meanings with more frequency, among which stand out, the derivative as the slope of the tangent line to the curve and the limit of the incremental quotient. The latter is used together with the procedural connection when the student used the power formula to obtain the derivative of a polynomial function ( $\frac{d(x^n)}{dx} = nx^{n-1}$ ), where the mathematical connection of feature type is also evidenced. These results are consistent with those reported by García-García and Dolores-Flores (2018) who identified this same mathematical connection

through the use of the power rule when finding the derivative and integral of polynomial functions in students at the pre-university level. Unlike the study by García-García and Dolores-Flores (2018), where the meaning of the derivative was identified as "the slope of the tangent line", in our investigation other meanings were identified: the derivative as the limit of the incremental quotient and the derivative as a reason for change. In addition, the results showed the connection between meanings of the derivative with the slope as the trigonometric tangent of the angle of inclination.

It was evident that S3, in his response, managed to make the six types of mathematical connections, while S1 made connections between different representations and implication. On the other hand, it was identified that S1, S2 and S3 made the five types of mathematical connections in the resolution of two tasks (tasks 3 and 5). However, S1 presented inconsistencies to solve the problem associated with the equation of the tangent line to a curve (Task 6), where he erroneously used the slope point formula leading to obtaining a wrong equation, that is, he did not make the procedural connection. The case studies of S2 and S3 showed they were able to consistently respond to tasks 6, 7 and 8, using the derivative concept appropriately by connecting it with other concepts and graphical representations. It should be noted that in response to task 8, S3 used reversibility relationships between  $f$  and  $f'$  since he sketched the graph of  $f$  from the information offered by  $f'$ . In this context, it was placed at a level 2 of connection quality since they are justifications and mathematically consistent processes that could promote the understanding of the derivative concept (Borji et al., 2018; Fuentealba et al., 2015, 2018a, 2018b).

It was found that students establish mathematical connections between multiple representations of the derivative, a situation that is difficult in the research of Sari et al. (2018) and in the study of Hashemi et al. (2014). We highlight the work done by S3 where he made mathematical connections of implication type (quality level 2) to graph  $f$  considering the information provided by  $f'$ . In contrast in Fuentealba et al. (2015), and Fuentealba et al. (2018a, 2018b) it was evidenced that students find it difficult to carry out reversibility processes, such as linking the signs of  $f'$  and  $f''$  to analyze the behavior of  $f$ . Also, in García-García and Dolores-Flores (2021a) the majority of students do not proceed visually to graph the antiderivative from the derivative (or vice versa), but they give priority to algebraic representations, so they established more mathematical connections of procedure and other students recognized the reversibility after having graphed. In Table 1 it is observed that in the type of tasks 7 and 8 the students managed to establish more mathematical connections, that is, the tasks that involve graphical aspects and application situations, allow us to analyze behaviours of the functions and establish

relationships between those derivatives from different orders (extreme values and concavity) favouring the conceptual understanding of the mathematical concept under study (Fuentealba et al., 2015).

It should be noted that in the from Campo-Meneses et al. (2021) research they do not consider the metaphorical connection because the study focused on the student's practice and not on the teacher's practice. However, in the present research some metaphorical connections that university students make in their mathematical activity when solving tasks on the derivative were identified. Therefore, in future research the category of metaphorical connections can be used in studies focused on both the student and the teacher of mathematics.

## CONCLUSION

The characterization of the quality of the mathematical connections allowed us to observe that university students use more graphical and algebraic processes, leaving aside formal arguments (theorems and definitions) that support the analytical processes in the resolution of a mathematical task. Likewise, we identified that those students who presented the difficulties to use the mathematical connection of different representations, were due to their limited conceptual knowledge about mathematical concepts, theorems and properties associated with the derivative, necessary to solve the proposed tasks. For this reason, in this study it was shown that the quality of the mathematical connections is determined based on the mathematical processes and arguments used by the student, considering that there is a connection when the answer is consistent with what is accepted mathematically; otherwise, we say that the student does not make mathematical connections. In this context, these non-established mathematical connections or quality level 0 are the main causes why students have difficulties to demonstrate a type of understanding about the derivative.

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