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Research Article

Mathematical Properties of the Hyperbolicity of Circulant Networks

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If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1 x_2]$, $[x_2 x_3]$, and $[x_3 x_1]$ in X. The space X is δ-*hyperbolic* (in the Gromov sense) if any side of T is contained in a δ-neighborhood of the union of the two other sides, for every geodesic triangle T in X. The study of the hyperbolicity constant in networks is usually a very difficult task; therefore, it is interesting to find bounds for particular classes of graphs. A network is circulant if it has a cyclic group of automorphisms that includes an automorphism taking any vertex to any other vertex. In this paper we obtain several sharp inequalities for the hyperbolicity constant of circulant networks; in some cases we characterize the graphs for which the equality is attained.

1. Introduction

The first works on Gromov hyperbolic spaces deal with finitely generated groups (see [1]). Initially, Gromov spaces were applied to the study of automatic groups in the science of computation (see, e.g., [2]); indeed, hyperbolic groups are strongly geodesically automatic; that is, there is an automatic structure on the group [3]. Besides, hierarchical networks have been found to have "hidden hyperbolic structure" [4]. For a study of other parameters in complex networks, see [5]. The concept of hyperbolicity appears also in discrete mathematics, algorithms, and networking. For example, it has been shown empirically in [6] that the Internet topology embeds with better accuracy into a hyperbolic space than into an Euclidean space of comparable dimension; the same holds for many complex networks; see [7]. A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [8]). Another important application of these spaces is the study of the spread of viruses on the Internet (see [9]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [9]). The study of Gromov hyperbolic networks is a subject of increasing interest (see, e.g., [7, 9–20] and the references therein).

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces (see [1, 21]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, Riemannian manifolds of negative sectional curvature bounded away from 0 and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1, 21]).

If $\gamma : [a, b] \to X$ is a continuous curve in a metric space (X, d), the *length* of γ is defined as

$$L(\gamma) := \sup \left\{ \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \cdots < t_n = b \right\}. \quad (1)$$

We say that γ is a *geodesic* if we have $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$ (then γ is equipped with an arc-length parametrization). The metric space X is said to be *geodesic* if for every couple of points in X there exists a geodesic joining them; we denote by $[xy]$ any geodesic joining x and y; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space X is a network, then the edge joining the vertices u and v will be denoted by $[u, v]$.

Along the paper we just consider graphs with every edge of length 1. In order to consider a network G as a geodesic metric space, identify (by an isometry) any edge $[u, v] \in E(G)$ with the interval $[0, 1]$ in the real line; then the edge $[u, v]$ (considered as a graph with just one edge) is isometric to the interval $[0, 1]$. Thus, the points in G are the vertices and, also, the points in the interior of any edge of G. In this way, any connected network G has a natural distance defined on its points, induced by taking the shortest paths in G, and we can see G as a metric graph. If x, y are in different connected components of G, we define $d_G(x, y) = \infty$. Throughout this paper, $G = (V, E)$ denotes a simple (without loops and multiple edges) graph (not necessarily connected) such that every edge has length 1 and $V \neq \emptyset$. These properties guarantee that any connected component of any network is a geodesic metric space. Note that to exclude multiple edges and loops is not an important loss of generality, since [13, Theorems 8 and 10] reduce the problem of computing the hyperbolicity constant of graphs with multiple edges and/or loops to the study of simple graphs.

If X is a geodesic metric space and $x_1, x_2, x_3 \in X$, the union of three geodesics $[x_1x_2]$, $[x_2x_3]$, and $[x_3x_1]$ is a *geodesic triangle* that will be denoted by $T = \{x_1, x_2, x_3\}$ and we will say that x_1, x_2 and x_3 are the vertices of T; it is usual to write also $T = \{[x_1x_2], [x_2x_3], [x_3x_1]\}$. We say that T is δ-*thin* if any side of T is contained in the δ-neighborhood of the union of the two other sides. We denote by $\delta(T)$ the sharp thin constant of T; that is, $\delta(T) := \inf\{\delta \geq 0 : T \text{ is } \delta\text{-thin}\}$. The space X is δ-*hyperbolic* (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ-thin. We denote by $\delta(X)$ the sharp hyperbolicity constant of X; that is, $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$. If we have a triangle with two identical vertices, we call it a *bigon*; note that since this is a special case of the definition, every geodesic bigon in a δ-hyperbolic space is δ-thin. We say that X is *hyperbolic* if X is δ-hyperbolic for some $\delta \geq 0$; then X is hyperbolic if and only if $\delta(X) < \infty$. If X has connected components $\{X_i\}_{i\in I}$, then we define $\delta(X) := \sup_{i\in I} \delta(X_i)$, and we say that X is hyperbolic if $\delta(X) < \infty$.

In the classical references on this subject (see, e.g., [1, 21]) several different definitions of Gromov hyperbolicity appear, which are equivalent in the sense that if X is δ-hyperbolic with respect to one definition, then it is δ'-hyperbolic with respect to another definition (for some δ' related to δ). The definition that we have chosen has a deep geometric meaning (see, e.g., [1, 21]).

Trivially, any bounded metric space X is $(\operatorname{diam} X)$-hyperbolic. A normed linear space is hyperbolic if and only if it has dimension one. A geodesic space is 0-hyperbolic if and only if it is a metric tree. If a complete Riemannian manifold is simply connected and its sectional curvatures satisfy $K \leq c$ for some negative constant c, then it is hyperbolic. See the classical reference [1, 21] in order to find further results.

A network is *circulant* if it has a cyclic group of automorphisms that includes an automorphism taking any vertex to any other vertex. There are large classes of circulant graphs. For instance, every cycle graph, complete graph, crown graph, and Möbius ladder is a circulant graph. A complete bipartite graph is a circulant graph if and only if it has the same number of vertices on both sides of its bipartition. A connected finite graph is circulant if and only if it is the Cayley graph of a cyclic group; see [22]. Every circulant graph is a vertex transitive graph and a Cayley graph [23].

The circulant is a natural generalization of the double loop network and was first considered by Wong and Coppersmith [24]. Our main interest in circulant graphs lies in the role they play in the design of networks. In the area of computer networks, the standard topology is that of a ring network, that is, a cycle in graph theoretic terms. However, cycles have relatively large diameter, and in an attempt to reduce the diameter by adding edges, we wish to retain certain properties. In particular, we would like to retain maximum connectivity and vertex-transitivity. Hence, most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems [25, 26]. The term circulant comes from the nature of its adjacency matrix. A matrix is circulant if all its rows are periodic rotations of the first one. Circulant matrices have been employed for designing binary codes [27]. Theoretical properties of circulant graphs have been studied extensively and surveyed [25].

The study of the hyperbolicity constant in networks is usually a very difficult task; therefore, it is interesting to find bounds of this constant for particular classes of graphs. For a general graph or a general geodesic metric space deciding whether or not a space is hyperbolic is usually very difficult. Therefore, it is interesting to relate the hyperbolicity with other classes of graphs. The papers [10, 14, 15, 20] prove, respectively, that chordal, k-chordal, edge-chordal, and join graphs are hyperbolic. Moreover, in [10] it is shown that hyperbolic graphs are path-chordal graphs. The authors have proved in a previous work that every circulant graph is hyperbolic (and they obtain inequalities for the hyperbolicity constant of infinite circulant graphs). In this paper we obtain several sharp inequalities for the hyperbolicity constant of finite circulant networks; in some cases we characterize the graphs for which the equality is attained. Theorem 3 in Section 2 gives the precise value of the hyperbolicity constant of $\delta(C_n(a_1))$. Theorem 11 provides a sharp lower bound for $\delta(C_n(a_1, a_2, \ldots, a_k))$ and characterizes the graphs for which the equality is attained. It is well known that a graph is circulant if and only if its complement is circulant. Thus it is natural to study in this context the properties of general complement graphs. In Theorems 15 and 24 this kind of results appears and they are applied to circulant graphs in Corollary 25. We collect in Section 3 several sharp inequalities for the hyperbolicity constant of a large class of circulant graphs. In Theorem 28 good lower and upper bounds for $\delta(C_n(1, a_2, \ldots, a_k))$ appear, which are improved in Theorems 29, 30, and 31 with additional hypothesis.

Furthermore, we obtain the precise value of the hyperbolicity constant of many circulant networks (see Theorems 3, 11, and 29 and Corollary 25).

2. Bounds for the Hyperbolicity Constant

Given any natural number $n \geq 3$, let $\{a_1, a_2, \ldots, a_k\}$ be a set of integers such that $0 < a_1 < \cdots < a_k \leq \lfloor n/2 \rfloor$, where $\lfloor t \rfloor$ denotes the lower integer part of t.

We define the circulant network $C_n(a_1, \ldots, a_k)$ as the finite graph with vertices $\{1, 2, \ldots, n\}$ (or $\{0, 1, \ldots, n-1\}$) such that $N(j) = \{j \pm a_i (\mathrm{mod} n)\}_{i=1}^{k}$ is the set of neighbors of each vertex j. If $a_k \neq n/2$, then $C_n(a_1, \ldots, a_k)$ is a regular graph of degree $2k$. For n even, we allow $a_k = n/2$; in this case, $C_n(a_1, \ldots, a_k)$ is regular of degree $2k - 1$.

The following result is well known (see, e.g., [28, Theorem 4.2]).

Theorem 1. *The circulant graph $C_n(a_1, \ldots, a_k)$ is connected if and only if* $\gcd(a_1, a_2, \ldots, a_k, n) = 1$.

If a circulant graph G has connected components $G_1, \ldots, G_k$, then G_i and G_j are isomorphic for every $1 \leq i, j \leq k$, G_j is also a circulant graph, and $\delta(G) = \delta(G_j)$ for every $1 \leq j \leq k$. Thus the condition G connected is not a real restriction (unless if we deal with the complement graph of G, as in Theorems 15 and 24 and Corollary 25).

As usual, by *cycle* we mean a simple closed curve, that is, a path with different vertices, unless the last one, which is equal to the first vertex.

We also need the following result in [18, Theorem 11].

Theorem 2. *If C_r is the cycle graph with $r \geq 3$ vertices, then* $\delta(C_r) = r/4$.

The next result provides the precise value of $\delta(C_n(a_1))$ for every value of n and a_1.

Theorem 3. *If $n = ja_1$ with $j \geq 2$, then the circulant graph $C_n(a_1)$ has a_1 connected components, $\delta(C_n(a_1)) = j/4$ for every $j \geq 3$ and $\delta(C_n(a_1)) = 0$ if $j = 2$.*

If $\gcd(n, a_1) = j < a_1$, then $C_n(a_1)$ has j connected components and $\delta(C_n(a_1)) = n/(4j)$.

Proof. If $n = ja_1$, then it is clear that $C_n(a_1)$ has a_1 connected components. If $j = 2$, then $C_n(a_1)$ is the disjoint union of a_1 edges and $\delta(C_n(a_1)) = 0$. If $j \geq 3$, then $C_n(a_1)$ is the disjoint union of a_1 graphs isomorphic to C_j. Thus Theorem 2 gives $\delta(C_n(a_1)) = j/4$.

If $\gcd(n, a_1) = j < a_1$, then $C_n(a_1)$ is the disjoint union of j graphs isomorphic to $C_{n/j}$, and Theorem 2 gives $\delta(C_n(a_1)) = n/(4j)$. □

From [17, Proposition 5 and Theorem 7] we deduce the following result.

Lemma 4. *Let G be any graph with a cycle g. If $L(g) \geq 3$, then $\delta(G) \geq 3/4$. If $L(g) \geq 4$, then $\delta(G) \geq 1$.*

For the sake of completeness, we are going to give an idea of the proof of this lemma. We need a definition and a lemma. We say that a subgraph Γ of G is *isometric* if $d_\Gamma(x, y) = d_G(x, y)$ for every $x, y \in \Gamma$. It is easy to check that a subgraph Γ of G is isometric if and only if $d_\Gamma(u, v) = d_G(u, v)$ for every $u, v \in V(\Gamma)$. Isometric subgraphs are very important in the study of hyperbolic graphs, as the following result shows.

Lemma 5 (see [18, Lemma 5]). *If Γ is an isometric subgraph of G, then $\delta(\Gamma) \leq \delta(G)$.*

Let us start with the idea of the proof of Lemma 4. If $L(g) = 3$, then g is an isometric subgraph, and Lemma 5 and Theorem 2 give $\delta(G) \geq \delta(g) = \delta(C_3) = 3/4$. If $L(g) = 4$, then the graph Γ induced by g is an isometric subgraph; thus, Γ is isomorphic to either C_4, K_4, or K_4 without an edge, and Lemma 5 gives $\delta(G) \geq \delta(\Gamma) = 1$. Assume now that $L(g) \geq 5$ and there is no cycle in G of length 4. Let g_0 be a curve with

$$L(g_0) = \min \{L(\gamma) \mid \gamma \text{ is a cycle in } G \text{ with } L(\gamma) \geq 5\}. \tag{2}$$

One can prove that g_0 is an isometric subgraph and Lemma 5 and Theorem 2 give $\delta(G) \geq \delta(g_0) = L(g_0)/4 \geq 5/4 > 1$.

By $x = \pm y$ we mean that we have either $x = y$ or $x = -y$.

Definition 6. Given $a_0 = 0$ and $1 \leq a_1 < a_2 < \cdots < a_k \leq \lfloor n/2 \rfloor$ one says that the sequence $\{a_1, a_2, \ldots, a_k\}$ is *n-full* if for every $0 \leq i \leq n-1$ and $1 \leq j \leq k$ there exists $0 \leq k_0 \leq k$ such that $i = \pm a_{k_0} (\mathrm{mod} n)$ or $i + a_j = \pm a_{k_0} (\mathrm{mod} n)$.

For any graph G, we define, as usual,

$$\operatorname{diam} V(G) := \sup \{d_G(v, w) \mid v, w \in V(G)\},$$
$$\operatorname{diam} G := \sup \{d_G(x, y) \mid x, y \in G\}. \tag{3}$$

Definition 7. One says that a vertex v of a graph G is a *cut-vertex* if $G \setminus \{v\}$ is not connected. A graph is *two-connected* if it does not contain cut-vertices. Given any edge in G, let one consider the maximal two-connected subgraph containing it. One calls to the set of these maximal two-connected subgraphs $\{G_s\}$ the *canonical T-decomposition* of G. One defines the effective diameter of G as

$$\operatorname{effdiam} V(G) := \sup_s \operatorname{diam} V(G_s),$$
$$\operatorname{effdiam} G := \sup_s \operatorname{diam} G_s. \tag{4}$$

Note that if G is a two-connected graph, then $\operatorname{effdiam} V(G) = \operatorname{diam} V(G)$ and $\operatorname{effdiam} G = \operatorname{diam} G$.

We need the following result in [12, Proposition 4.5 and Theorem 4.14].

Theorem 8. *A graph G verifies $\delta(G) = 1$ if and only if* $\operatorname{effdiam} G = 2$. *Furthermore, $\delta(G) \leq 1$ if and only if* $\operatorname{effdiam} G \leq 2$.

We need the following result in [18, Theorem 8].

Theorem 9. *In any graph G the inequality* $\delta(G) \le (\operatorname{diam} G)/2$ *holds.*

We have the following direct consequence.

Corollary 10. *In any graph G the inequality* $\delta(G) \le (\operatorname{diam} V(G) + 1)/2$ *holds.*

Denote by $J(G)$ the set of vertices and midpoints of edges in G.

Since Theorem 3 gives the precise value of $\delta(C_n(a_1))$, in order to study $\delta(C_n(a_1, a_2, \ldots, a_k))$ we just need to deal with the case $k > 1$.

We prove now a sharp lower bound for the hyperbolicity constant and we characterize the graphs for which this lower bound is attained.

Theorem 11. *For any integers* $k > 1$ *and* $1 \le a_1 < a_2 < \cdots < a_k \le \lfloor n/2 \rfloor$ *such that* $C_n(a_1, a_2, \ldots, a_k)$ *is connected, one has*

$$\delta\left(C_n\left(a_1, a_2, \ldots, a_k\right)\right) \ge 1, \quad (5)$$

and $\delta(C_n(a_1, a_2, \ldots, a_k)) = 1$ *if and only if* $\{a_1, a_2, \ldots, a_k\}$ *is n-full.*

Proof. We are going to prove that $C_n(a_1, a_2, \ldots, a_k)$ contains a cycle with length at least 4.

Assume first that $\operatorname{lcm}(a_j, n)/a_j \ge 4$ for some $1 \le j \le k$. Thus, $C_n(a_1, a_2, \ldots, a_k)$ contains a cycle with length at least 4.

Assume now that $\operatorname{lcm}(a_j, n)/a_j \le 3$ for every $1 \le j \le k$. Seeking for a contradiction assume that $\operatorname{lcm}(a_j, n)/a_j = 1$ for some $1 \le j \le k$. Then $\operatorname{lcm}(a_j, n) = a_j$ and $n \le a_j$, contradicting $a_j \le \lfloor n/2 \rfloor$. So, $2 \le \operatorname{lcm}(a_j, n)/a_j \le 3$ for every $1 \le j \le k$. If $\operatorname{lcm}(a_j, n)/a_j = 2$ for some $1 \le j \le k$, then $a_j = n/2 = a_k$. If $\operatorname{lcm}(a_j, n)/a_j = 3$ for some $1 \le j \le k$, then $a_j = n/3$. Since $k > 1$, we deduce $k = 2$, $a_1 = n/3$, and $a_2 = n/2$, and there exists a positive integer n_0 such that $n = 6n_0$, $a_1 = 2n_0$, and $a_2 = 3n_0$. Since $C_n(a_1, a_2, \ldots, a_k)$ is connected, Theorem 1 gives that $1 = \gcd(a_1, a_2, n) = \gcd(2n_0, 3n_0, 6n_0) = n_0$. Hence, $C_{6n_0}(2n_0, 3n_0) = C_6(2, 3)$, and the cycle with consecutive vertices $\{0, 2, 4, 1, 5, 3, 0\}$ has length $6 \ge 4$.

Thus, $C_n(a_1, a_2, \ldots, a_k)$ contains a cycle with length at least 4 in any case, and Lemma 4 gives $\delta(C_n(a_1, a_2, \ldots, a_k)) \ge 1$.

Denote by $\{0, 1, \ldots, n-1\}$ the vertices of $G := C_n(a_1, a_2, \ldots, a_k)$.

Assume first that $\{a_1, a_2, \ldots, a_k\}$ is n-full. We are going to show that $d(x, y) \le 3/2$ for every $x \in V(G)$ and $y \in J(G) \setminus V(G)$. Since G is a circulant graph, we can assume $x = 0$ by symmetry. Since $y \in J(G) \setminus V(G)$, there exist $0 \le i \le n-1$ and $1 \le j \le k$ such that y belongs to $[i, i+a_j]$. Thus there exists $0 \le k_0 \le k$ such that $i = \pm a_{k_0} (\operatorname{mod} n)$ or $i + a_j = \pm a_{k_0} (\operatorname{mod} n)$, and we have $d(0, [i, i+a_j]) \le 1$. Hence, $d(0, y) = d(0, [i, i+a_j]) + 1/2 \le 1 + 1/2 = 3/2$. Therefore, $d(x, y) \le 3/2$ for every $x \in V(G)$ and $y \in J(G) \setminus V(G)$, and we conclude $d(x, y) \le 2$ for every $x, y \in V(G)$ and for every $x, y \in J(G) \setminus V(G)$. Thus $\operatorname{diam} G \le 2$ and Theorem 9 gives $\delta(G) \le 1$, and we conclude $\delta(G) = 1$.

Assume now that $\delta(G) = 1$. Since G is a two-connected graph, Theorem 8 gives $\operatorname{diam} G = 2$. Hence, $d(x, y) \le 3/2$ for every $x \in V(G)$ and $y \in J(G) \setminus V(G)$. Consider $0 \le i \le n-1$ and $1 \le j \le k$ and let y be the midpoint of $[i, i+a_j]$. Therefore, $d(0, y) \le 3/2$ implies $d(0, [i, i+a_j]) \le 1$. Thus there exists $0 \le k_0 \le k$ such that $i = \pm a_{k_0} (\operatorname{mod} n)$ or $i + a_j = \pm a_{k_0} (\operatorname{mod} n)$, and we conclude that $\{a_1, a_2, \ldots, a_k\}$ is n-full. □

In [11, Theorem 2.6] the following result appears.

Theorem 12. *For every hyperbolic graph G,* $\delta(G)$ *is a multiple of* $1/4$.

Theorems 11 and 12 have the following consequences.

Corollary 13. *For any integers* $k > 1$ *and* $1 \le a_1 < a_2 < \cdots < a_k \le \lfloor n/2 \rfloor$ *such that* $C_n(a_1, a_2, \ldots, a_k)$ *is connected, one has*

$$\delta\left(C_n\left(a_1, a_2, \ldots, a_k\right)\right) \ge \frac{5}{4} \quad (6)$$

if and only if $\{a_1, a_2, \ldots, a_k\}$ *is not n-full.*

Corollary 14. *For any integers* $4 \le n \le 6$, $k > 1$, *and* $1 < a_2 < \cdots < a_k \le \lfloor n/2 \rfloor$, *one has*

$$\delta\left(C_n\left(1, a_2, \ldots, a_k\right)\right) = 1, \quad (7)$$

and $\delta(C_6(2, 3)) = 5/4$.

Proof. If we have either $n = 4$ or $n = 5$, then $k = 2 = \lfloor n/2 \rfloor$ and $\{a_1, a_2\} = \{1, 2\}$, and so $\operatorname{diam} V(C_n(1, 2)) = 1$ and Corollary 10 gives $\delta(C_n(1, 2)) \le 1$. Theorem 11 gives the converse inequality.

Assume that $n = 6$. If $k = 3 = \lfloor n/2 \rfloor$, then $\{a_1, a_2, a_3\} = \{1, 2, 3\}$, and the previous argument gives $\delta(C_6(1, 2, 3)) = 1$. If $k = 2$, then we have either $a_2 = 2$ or $a_2 = 3$. If $a_2 = 2$, then we have $0 = a_0$, $1 = a_1$, $2 = a_2$, $4 = -2 (\operatorname{mod} 6) = -a_2 (\operatorname{mod} 6)$, $5 = -1 (\operatorname{mod} 6) = -a_1 (\operatorname{mod} 6)$, $3 + 1 = 4 = -2 (\operatorname{mod} 6) = -a_2 (\operatorname{mod} 6)$, and $3 + 2 = 5 = -1 (\operatorname{mod} 6) = -a_1 (\operatorname{mod} 6)$, and Theorem 11 gives $\delta(C_6(1, 2)) = 1$. If $a_2 = 3$, then we have $0 = a_0$, $1 = a_1$, $3 = a_2$, $5 = -1 (\operatorname{mod} 6) = -a_1 (\operatorname{mod} 6)$, $2 + 1 = 3 = a_2$, $2 + 3 = 5 = -1 (\operatorname{mod} 6) = -a_1 (\operatorname{mod} 6)$, $4 + 1 = 5 = -1 (\operatorname{mod} 6) = -a_1 (\operatorname{mod} 6)$, and $4 + 3 = 7 = 1 (\operatorname{mod} 6)$, and Theorem 11 gives $\delta(C_6(1, 3)) = 1$.

Finally, consider $C_6(2, 3)$, $i = 5$, $j = 1$, and $i + a_j = 5 + 2 = 1 (\operatorname{mod} 6)$. Since $1 \ne \pm 2 (\operatorname{mod} 6)$, $1 \ne \pm 3 (\operatorname{mod} 6)$, $5 \ne \pm 2 (\operatorname{mod} 6)$, and $5 \ne \pm 2 (\operatorname{mod} 6)$, Corollary 13 gives $\delta(C_6(2, 3)) \ge 5/4$. One can check that $\operatorname{diam} C_6(2, 3) = 5/2$, and Theorem 9 gives $\delta(C_6(2, 3)) \le 5/4$. □

It is well known that a graph is circulant if and only if its complement is circulant. Thus it is natural to study in this context the properties of general complement graphs. In Theorems 15 and 24 this kind of results appears and they are applied to circulant graphs in Corollary 25.

As usual, the *complement* $\overline{G}$ of the (connected or nonconnected) graph G is defined as the graph with $V(\overline{G}) = V(G)$ such that $e \in E(\overline{G})$ if and only if $e \notin E(G)$.

Theorem 15. *If G is a graph with* $\operatorname{diam} V(G) \ge 4$, *then* $\overline{G}$ *is connected and* $\delta(\overline{G}) \le 3/2$, *and this inequality is sharp.*

Proof. Seeking for a contradiction assume that there exists an edge $e \in E(G)$ such that $d_G(e,v) \le 1$ for every $v \in V(G)$. Choose $u, v \in V(G)$ with $d_G(u,v) = 4$. Thus $4 = d_G(u,v) \le d_G(u,e) + L(e) + d_G(e,v) \le 3$, which is a contradiction. Hence, for each edge $e \in E(G)$ there exists $v \in V(G)$ with $d_G(e,v) \ge 2$.

Fix $w_1, w_2 \in V(G)$. If $d_G(w_1,w_2) > 1$, then $[w_1,w_2] \notin E(G)$, $[w_1,w_2] \in E(\overline{G})$, and $d_{\overline{G}}(w_1,w_2) = 1$. If $d_G(w_1,w_2) = 1$, then there exists $v \in V(G)$ with $d_G([w_1,w_2],v) \ge 2$. Thus $[w_1,v],[w_2,v] \notin E(G)$, $[w_1,v],[w_2,v] \in E(\overline{G})$, and $d_{\overline{G}}(w_1,w_2) \le d_{\overline{G}}(w_1,v) + d_{\overline{G}}(v,w_2) = 2$. Hence, $\overline{G}$ is connected, $\operatorname{diam} V(\overline{G}) \le 2$, and Corollary 10 gives the inequality.

The following family of graphs shows that this inequality is sharp. Let $n \ge 912$ be an even integer and $G = C_n(2,6,8,n/2-2,n/2-1)$. Since G is a 10-regular graph and $n \ge 912 > 911 = 1 + 10(1 + 9 + 9^2)$, the Moore's bound gives $\operatorname{diam} V(G) \ge 4$. Hence, we have proved that $\delta(\overline{G}) \le 3/2$ and it suffices to show that $\delta(\overline{G}) \ge 3/2$. Denote by $\{0,1,\ldots,n-1\}$ the vertices of G and consider the cycle C in $\overline{G}$ with length 6 and consecutive vertices $\{1,2,n-1,n-4,n/2,4,1\}$. Let x, y be the midpoints of $[1,4]$ and $[n-4,n-1]$, respectively, and γ_1, γ_2 the two geodesics contained in C joining x and y with vertices $\{4,n/2,n-4\}$ and $\{1,2,n-1\}$, respectively. Since $[n/2,1],[n/2,2],[n/2,n-1] \in E(G)$, we have $[n/2,1],[n/2,2],[n/2,n-1] \notin E(\overline{G})$ and

$$\delta\left(\overline{G}\right) \ge d_{\overline{G}}\left(\frac{n}{2},\gamma_2\right) = d_{\overline{G}}\left(\frac{n}{2},\{x,y\}\right) = \frac{3}{2}. \tag{8}$$

□

Theorem 24 below gives more information than Theorem 15 for nonconnected graphs. In order to prove it, we need some technical results.

Lemma 16. *If G is a nonconnected graph with connected components $G_1,\ldots,G_k$ and $d_{\overline{G_j}}(v,e) \le 1$ for every $v \in V(\overline{G_j})$, $e \in E(\overline{G_j})$, and $1 \le j \le k$, then* $\operatorname{diam}\overline{G} \le 2$.

Proof. Note that it suffices to check that $d_{\overline{G}}(x,v) \le 3/2$ for every $x \in J(\overline{G}) \setminus V(\overline{G})$ and $v \in V(\overline{G})$.

Let x be the midpoint of $[v',v''] \in E(\overline{G_j})$ for some $1 \le j \le k$. If $v \in V(\overline{G_j})$, then $d_{\overline{G}}(x,v) \le 3/2$ since $\operatorname{diam}\overline{G_j} \le 2$. If $v \in V(\overline{G_i})$ for some $i \ne j$, then $d_{\overline{G}}(x,v) \le d_{\overline{G}}(x,v') + d_{\overline{G}}(v',v) = 3/2$.

Let x be the midpoint of $[v_i,v_j] \in E(\overline{G})$ with $v_i \in V(\overline{G_i})$, $v_j \in V(\overline{G_j})$, and $i \ne j$. If $v \in V(\overline{G_i})$, then $d_{\overline{G}}(x,v) \le d_{\overline{G}}(x,v_j) + d_{\overline{G}}(v_j,v) = 3/2$. If $v \notin V(\overline{G_i})$, then $d_{\overline{G}}(x,v) \le d_{\overline{G}}(x,v_i) + d_{\overline{G}}(v_i,v) = 3/2$.

Hence, $d_{\overline{G}}(x,v) \le 3/2$ for every $x \in J(\overline{G}) \setminus V(\overline{G})$ and $v \in V(\overline{G})$, and we conclude $\operatorname{diam}\overline{G} \le 2$. □

Note that a connected graph Γ satisfies $d_\Gamma(v,e) \le 1$ for every $v \in V(\Gamma)$, $e \in E(\Gamma)$ if and only if $\operatorname{diam}\Gamma \le 2$. A nonconnected graph Γ satisfies this property if and only if $E(\Gamma) = \emptyset$.

We have the following direct consequence of Lemma 16.

Corollary 17. *If G is a nonconnected graph with connected components $G_1,\ldots,G_k$ and* $\operatorname{diam}\overline{G_j} \le 2$ *for every $1 \le j \le k$, then* $\operatorname{diam}\overline{G} \le 2$.

Let $G = (V(G),E(G))$ and $\Gamma = (V(\Gamma),E(\Gamma))$ be two graphs with $V(G) \cap V(\Gamma) = \emptyset$. We recall that the *graph join* $G \uplus \Gamma$ of G and Γ is the graph such that $V(G \uplus \Gamma) = V(G) \cup V(\Gamma)$ and two different vertices u and v of $G \uplus \Gamma$ are adjacent if $u \in V(G)$ and $v \in V(\Gamma)$, or $[u,v] \in E(G)$ or $[u,v] \in E(\Gamma)$.

The argument in the proof of Lemma 16 gives the following result.

Corollary 18. *If G and Γ are graphs with* $\operatorname{diam} G \le 2$ *and* $\operatorname{diam}\Gamma \le 2$, *then* $\operatorname{diam}(G \uplus \Gamma) \le 2$.

Definition 19. One says that a nonconnected graph G with connected components $G_1,\ldots,G_k$ satisfies the 1*-vertex-edge property* if we have either $d_{\overline{G_j}}(v,e) \le 1$, for every $v \in V(\overline{G_j})$, $e \in E(\overline{G_j})$, and $1 \le j \le k$, or $k = 2$, $|V(G_1)| = 1$, and $\operatorname{diam}\Gamma_i \le 2$, for every $1 \le i \le r$, where $\Gamma_1,\ldots,\Gamma_r$ $(r \ge 1)$ are the connected components of $\overline{G_2}$, or $k = 2$, $|V(G_2)| = 1$, and $\operatorname{diam}\Gamma_i \le 2$ for every $1 \le i \le r$ where $\Gamma_1,\ldots,\Gamma_r$ $(r \ge 1)$ are the connected components of $\overline{G_1}$.

Theorem 20. *Let G be a nonconnected graph with connected components $G_1,\ldots,G_k$. Then* $\operatorname{effdiam}\overline{G} \le 2$ *if and only if G satisfies the* 1*-vertex-edge property.*

Proof. Assume that $d_{\overline{G_j}}(v,e) \le 1$ for every $v \in V(\overline{G_j})$, $e \in E(\overline{G_j})$, and $1 \le j \le k$. Lemma 16 gives $\operatorname{effdiam}\overline{G} \le \operatorname{diam}\overline{G} \le 2$.

Assume now that $k = 2$, $|V(G_1)| = 1$, and $\operatorname{diam}\Gamma_i \le 2$ for every $1 \le i \le r$ where $\Gamma_1,\ldots,\Gamma_r$ $(r \ge 1)$ are the connected components of $\overline{G_2}$. If $V(G_1) = \{v\}$, then we define for each $1 \le i \le r$ the graph $\Gamma_i' = \{v\} \uplus \Gamma_i$. Corollary 18 gives that $\operatorname{diam}\Gamma_i' \le 2$. Since $\{\Gamma_i'\}$ is the canonical T-decomposition of $\overline{G}$, we conclude that $\operatorname{effdiam}\overline{G} = \max_{1\le i\le r}\operatorname{diam}\Gamma_i' \le 2$.

If $k = 2$, $|V(G_2)| = 1$, and $\operatorname{diam}\Gamma_i \le 2$ for every $1 \le i \le r$ where $\Gamma_1,\ldots,\Gamma_r$ $(r \ge 1)$ are the connected components of $\overline{G_1}$, then the previous argument also gives $\operatorname{effdiam}\overline{G} \le 2$.

Finally, assume that $\operatorname{effdiam}\overline{G} \le 2$.

Note that $\overline{G}$ has a cut-vertex if and only if $k = 2$ and we have either $|V(G_1)| = 1$ and $\overline{G_2}$ being nonconnected or $|V(G_2)| = 1$ and $\overline{G_1}$ being nonconnected.

Assume that $\overline{G}$ has a cut-vertex. By symmetry we can assume that $k = 2$, $|V(G_1)| = 1$, and $\overline{G_2}$ is nonconnected. Let $\Gamma_1,\ldots,\Gamma_r$ $(r \ge 2)$ be the connected components of $\overline{G_2}$, and consider $\Gamma_1',\ldots,\Gamma_r'$ defined as before. Thus $\operatorname{diam}\Gamma_i' \le \operatorname{effdiam}\overline{G} \le 2$ for every $1 \le i \le r$. Seeking for a contradiction assume that $\operatorname{diam}\Gamma_i > 2$ for some $1 \le i \le r$. Then $d_{\Gamma_i}(x,v) = 5/2$ for some $x \in J(\Gamma_i) \setminus V(\Gamma_i')$ and $v \in V(\Gamma_i)$, and we conclude $d_{\Gamma_i'}(x,v) = 5/2$, which is a contradiction. Hence, $\operatorname{diam}\Gamma_i \le 2$ for every $1 \le i \le r$, and G satisfies the 1-vertex-edge property.

Assume now that $\overline{G}$ does not have cut-vertices. Thus $\operatorname{diam}\overline{G} = \operatorname{effdiam}\overline{G} \le 2$. Seeking for a contradiction assume that $d_{\overline{G_j}}(v,e) \geq 2$ for some $v \in V(\overline{G_j})$, $e \in E(\overline{G_j})$, and $1 \leq j \leq k$. If x is the midpoint of e, then $d_{\overline{G_j}}(x,v) \geq 5/2$, and we conclude $d_{\overline{G}}(x,v) = 5/2$, which is a contradiction. Hence, $d_{\overline{G_j}}(v,e) \leq 1$ for every $v \in V(\overline{G_j})$, $e \in E(\overline{G_j})$, and $1 \leq j \leq k$, and G satisfies the 1-vertex-edge property. □

Consider the set $\mathbb{T}_1$ of geodesic triangles T in G that are cycles such that the three vertices of the triangle T belong to $J(G)$.

The following result appears in [11, Theorem 2.7].

Theorem 21. *For any hyperbolic graph G there exists a geodesic triangle $T \in \mathbb{T}_1$ such that $\delta(T) = \delta(G)$.*

The following result in [17, Theorem 11] will be useful.

Theorem 22. *If G is a graph with $\delta(G) < 1$, then one has either $\delta(G) = 0$ or $\delta(G) = 3/4$. Furthermore,*

(i) $\delta(G) = 0$ *if and only if G is a tree;*

(ii) $\delta(G) = 3/4$ *if and only if $\delta(G) > 0$ and every cycle g in G has length $L(g) = 3$.*

Definition 23. Given a graph G with $\operatorname{diam} V(G) = 2$, one says that a subgraph G_0 contains a *maximal triangle* and there exists a geodesic triangle T in G that is a cycle such that $x, y, z \in J(G)$, $\delta(T) = 3/2$, and T is contained in G_0.

Note that if G_0 contains a maximal triangle T, then we can rename the vertices of T in order to guarantee that there exists $p \in [xy]$ such that $d_G(p,[xz]\cup[zy]) = d_G(p,x) = d_G(p,y) = L([xy])/2 = \delta(T) = 3/2$, $x, y \in J(G) \setminus V(G)$, and $p \in V(G)$. Furthermore, $L([yz]) \leq 3$ and $L([xz]) \leq 3$.

The following result provides the precise value of $\delta(\overline{G})$ for every nonconnected graph G.

Theorem 24. *If G is a nonconnected graph with connected components $G_1, \ldots, G_k$, then $\overline{G}$ is connected and $\delta(\overline{G}) \leq 3/2$. Furthermore,*

(i) $\delta(\overline{G}) = 0$ *if and only if $k = 2$ and G_1 and G_2 are complete graphs and we have $|V(G_1)| = 1$ or $|V(G_2)| = 1$;*

(ii) $\delta(\overline{G}) = 3/4$ *if and only if $\delta(\overline{G}) > 0$ and we have either $k = 3$, $|V(G_1)| = |V(G_2)| = |V(G_3)| = 1$, or $k = 2$, $|V(G_1)| = 1$, and G_2 being isomorphic to a complete graph without a nonempty set of pairwise disjoint edges, or $k = 2$, $|V(G_2)| = 1$, and G_1 being isomorphic to a complete graph without a nonempty set of pairwise disjoint edges;*

(iii) $\delta(\overline{G}) = 1$ *if and only if $\delta(\overline{G}) > 3/4$ and G satisfies the 1-vertex-edge property;*

(iv) $\delta(\overline{G}) = 5/4$ *if and only if $\delta(\overline{G}) > 1$ and $\overline{G_j}$ does not contain a maximal triangle for every $1 \leq j \leq k$;*

(v) $\delta(\overline{G}) = 3/2$ *if and only if $\overline{G_j}$ contains a maximal triangle for some $1 \leq j \leq k$.*

Proof. Theorem 15 gives that $\overline{G}$ is connected and $\delta(\overline{G}) \leq 3/2$. Furthermore, the argument in the proof of Theorem 15 provides that $\operatorname{diam} V(\overline{G}) \leq 2$ and thus $\operatorname{diam} \overline{G} \leq 3$.

$\overline{G}$ is a tree if and only if $k = 2$ and G_1 and G_2 are complete graphs and we have $|V(G_1)| = 1$ or $|V(G_2)| = 1$. This gives the first item.

$\overline{G}$ is not a tree and every cycle g in $\overline{G}$ has length $L(g) = 3$ if and only if we have either $k = 3$, $|V(G_1)| = |V(G_2)| = |V(G_3)| = 1$ (if $\overline{G}$ does not have cut-vertices), or $k = 2$, $|V(G_1)| = 1$, and G_2 being isomorphic to a complete graph without a nonempty set of pairwise disjoint edges, or $k = 2$, $|V(G_2)| = 1$, and G_1 being isomorphic to a complete graph without a nonempty set of pairwise disjoint edges (if $\overline{G}$ has a cut-vertex). Thus Theorem 22 gives the second item.

Assume that $\delta(\overline{G}) = 3/2$. By Theorem 9, we have $\operatorname{diam} \overline{G} = 3$ and $\operatorname{diam} V(\overline{G}) = 2$. By Theorem 21, there exist a geodesic triangle $T = \{x, y, z\}$ that is a cycle in $\overline{G}$ and $p \in [xy]$ such that $d_{\overline{G}}(p, [xz] \cup [zy]) = \delta(T) = \delta(\overline{G}) = 3/2$ and $x, y, z \in J(\overline{G})$. Since $\operatorname{diam} \overline{G} = 3$ and $\operatorname{diam} V(\overline{G}) = 2$, we have $L([xy])/2 = d_{\overline{G}}(p,x) = d_{\overline{G}}(p,y) = d_{\overline{G}}(p,[xz]\cup[zy]) = \delta(T) = \delta(\overline{G}) = 3/2$ and p is the midpoint of $[xy]$. Thus $x, y \in J(\overline{G}) \setminus V(\overline{G})$ and $p \in V(\overline{G})$. Besides, $L([yz]) \leq 3$ and $L([xz]) \leq 3$.

Let q be the midpoint of $[v_i, v_j] \in E(\overline{G})$ with $v_i \in V(\overline{G_i})$, $v_j \in V(\overline{G_j})$, and $i \neq j$. If $v \in V(\overline{G_i})$, then $d_{\overline{G}}(q,v) \leq d_{\overline{G}}(q,v_j) + d_{\overline{G}}(v_j,v) = 3/2$. If $v \notin V(\overline{G_i})$, then $d_{\overline{G}}(q,v) \leq d_{\overline{G}}(q,v_i) + d_{\overline{G}}(v_i,v) = 3/2$. Hence, $d_{\overline{G}}(q,v) \leq 3/2$ for every $v \in V(\overline{G})$ and $d_{\overline{G}}(q,w) \leq 2$ for every $w \in J(\overline{G})$.

Let q be the midpoint of $[v', v''] \in E(\overline{G_j})$ for some $1 \leq j \leq k$ and $v \in V(\overline{G_i})$ for some $i \neq j$. Thus, $d_{\overline{G}}(q,v) = 3/2$ and $d_{\overline{G}}(q,w) \leq 2$ for every $w \in J(\overline{G}) \setminus J(\overline{G_j})$.

Therefore, there exist $1 \leq j \leq k$ and $[v', v''], [w', w''] \in E(\overline{G_j})$ such that x and y are the midpoints of $[v', v'']$ and $[w', w'']$, respectively. By symmetry, we can assume that $v', w' \in [xy]$, and so $V(\overline{G}) \cap [xy] = \{v', p, w'\}$. Since T is a cycle, $v'', w'' \in [xz] \cup [zy]$. If $p \notin V(\overline{G_j})$, then $3/2 = d_{\overline{G}}(p, [xz] \cup [zy]) \leq d_{\overline{G}}(p, v'') = 1$, which is a contradiction. Hence, $p \in V(\overline{G_j})$. If there exists a vertex $v \in V(\overline{G}) \cap ([xz] \cup [zy])$ with $v \notin V(\overline{G_j})$, then $3/2 = d_{\overline{G}}(p, [xz] \cup [zy]) \leq d_{\overline{G}}(p, v) = 1$, which is a contradiction. Therefore, $T \cap V(\overline{G}) \subseteq \overline{G_j}$ and so T is a maximal triangle in $\overline{G_j}$.

Assume now that $\overline{G_j}$ contains a maximal triangle T for some $1 \leq j \leq k$. Thus $3/2 = \delta(T) \leq \delta(\overline{G})$ and, since $\delta(\overline{G}) \leq 3/2$, we conclude $\delta(\overline{G}) = 3/2$.

Theorem 20 gives that $\operatorname{effdiam} \overline{G} \leq 2$ if and only if G satisfies the 1-vertex-edge property. By Theorem 8, $\delta(\overline{G}) \leq 1$ if and only if G satisfies the 1-vertex-edge property. Theorem 12 gives $\delta(\overline{G}) > 3/4$ if and only if $\delta(\overline{G}) \geq 1$. Hence, $\delta(\overline{G}) = 1$ if and only if $\delta(\overline{G}) > 3/4$ and G satisfies the 1-vertex-edge property.

Finally, the previous results and Theorem 12 provide the characterization of the graphs G with $\delta(\overline{G}) = 5/4$. □

Theorem 24 has the following consequence for circulant graphs.

Corollary 25. *Fix integers $k \ge 1$ and $1 \le a_1 < a_2 < \cdots < a_k \le \lfloor n/2 \rfloor$ such that $G := C_n(a_1, a_2, \dots, a_k)$ is nonconnected, and consider integers $r > 1$ and $1 \le b_1 < b_2 < \cdots < b_k \le \lfloor n/(2r) \rfloor$ such that $C_n(a_1, a_2, \dots, a_k)$ has r connected components isomorphic to $C_{n/r}(b_1, b_2, \dots, b_k)$. Then $\overline{G}$ is a connected circulant graph and $1 \le \delta(\overline{G}) \le 3/2$. Furthermore,*

(i) $\delta(\overline{G}) = 1$ *if and only if we have either* $\operatorname{diam} \overline{C_{n/r}(b_1, b_2, \dots, b_k)} \le 2$ *or* $\overline{C_{n/r}(b_1, b_2, \dots, b_k)}$ *being a complete graph;*

(ii) $\delta(\overline{G}) = 5/4$ *if and only if* $\delta(\overline{G}) > 1$ *and* $\overline{C_{n/r}(b_1, b_2, \dots, b_k)}$ *does not contain a maximal triangle;*

(iii) $\delta(\overline{G}) = 3/2$ *if and only if* $\overline{C_{n/r}(b_1, b_2, \dots, b_k)}$ *contains a maximal triangle.*

Proof. Since $1 \le \lfloor n/(2r) \rfloor$ and $|V(C_{n/r}(b_1, b_2, \dots, b_k))| = n/r \ge 2 > 1$. Hence, G satisfies the 1-vertex-edge property if and only if we have either $\operatorname{diam} \overline{C_{n/r}(b_1, b_2, \dots, b_k)} \le 2$ or $\overline{C_{n/r}(b_1, b_2, \dots, b_k)}$ being the complete graph with n/r vertices. Thus Theorem 24 gives the result. □

3. Bounds for the Hyperbolicity Constant If $a_1 = 1$

The following result is well known (see, e.g., [28, Proposition 5.1]).

Theorem 26. *If $C_n(a_1, a_2, \dots, a_k)$ is such that $\gcd(n, a_i) = 1$ for some i with $1 \le i \le k$, then there exists a circulant graph $C_n(b_1, b_2, \dots, b_k)$ isomorphic to $C_n(a_1, a_2, \dots, a_k)$ with $b_1 = 1$.*

Hence, it is natural to find bounds for the hyperbolicity constant of $C_n(1, a_2, \dots, a_k)$. We will need the following result.

If H is a subgraph of G and $w \in V(H)$, we denote by $\deg_H(w)$ the degree of the vertex w in the subgraph induced by $V(H)$.

Theorem 27 (see [12, Theorem 3.2]). *Let G be any graph. Then $\delta(G) \ge 5/4$ if and only if there exist a cycle g in G with length $L(g) \ge 5$ and a vertex $w \in g$ such that $\deg_g(w) = 2$.*

The following result provides good lower and upper bounds if $a_1 = 1$.

Theorem 28. *For any integers $k > 1$ and $1 < a_2 < \cdots < a_k \le \lfloor n/2 \rfloor$ one has*

$$\frac{1}{2}\left\lfloor \frac{n}{2a_k} \right\rfloor \le \delta\left(C_n\left(1, a_2, \dots, a_k\right)\right) \le \frac{1}{2}\left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k + 1}{4}, \quad (9)$$

if $n - 2a_k \lfloor n/(2a_k) \rfloor \in \{a_k + 1, a_k + 2, 2a_k - 1\}$, and

$$\frac{1}{2}\left\lfloor \frac{n}{2a_k} \right\rfloor \le \delta\left(C_n\left(1, a_2, \dots, a_k\right)\right) \le \frac{1}{2}\left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k}{4}, \quad (10)$$

otherwise. The second equality in (9) is attained if $k = 2$, a_2 is odd, and $n - 2a_2 \lfloor n/(2a_2) \rfloor = a_2 + 1$. The second equality in (10) is attained if $k = 2$ and n is an odd multiple of a_2.

Proof. Let us denote by $\{0, 1, \dots, n-1\}$ the vertices of $G := C_n(1, a_2, \dots, a_k)$, and let us denote by C_n the subgraph of G with $V(C_n) = V(G)$ and $E(C_n) = \{[0,1], [1,2], \dots, [n-2, n-1], [n-1, 1]\}$.

We prove first the upper bounds.

We are going to find an upper bound of $\operatorname{diam} G$. We want to remark that it is not possible to find a simple formula for $\operatorname{diam} G$ (and not even for $\operatorname{diam} V(G)$, see [28]).

Fix a vertex $v \in V(G)$, and denote by v', v'' the vertices with $d_{C_n}(v, v') = d_{C_n}(v, v'') = a_k \lfloor n/(2a_k) \rfloor$ (if n is a multiple of $2a_k$, then $v' = v''$); therefore, $d_{C_n}(v', v'') = n - 2a_k \lfloor n/(2a_k) \rfloor$ and $d_G(v, v') = d_G(v, v'') = \lfloor n/(2a_k) \rfloor$. For each real number t with $0 \le t \le d_{C_n}(v', v'') \le 2a_k - 1$, define v_t as the point in C_n with $d_{C_n}(v_t, v') = t$ and $d_{C_n}(v_t, v) \ge a_k \lfloor n/(2a_k) \rfloor$.

Assume that $0 < d_{C_n}(v', v'') \le a_k - 1$ (the case $d_{C_n}(v', v'') = 0$ is trivial).

We have

$$\begin{aligned} d_G\left(v, v_t\right) &\le \left\lfloor \frac{n}{2a_k} \right\rfloor + \min\left\{t, d_{C_n}\left(v', v''\right) - t\right\} \\ &\le \left\lfloor \frac{n}{2a_k} \right\rfloor + \min\left\{t, a_k - 1 - t\right\} \\ &\le \left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k - 1}{2}. \end{aligned} \quad (11)$$

If $d_{C_n}(v', v'') > a_k$, then we define v''' as the vertex verifying $d_{C_n}(v''', v'') = a_k$ and $d_{C_n}(v''', v') = d_{C_n}(v', v'') - a_k$. Assume that $a_k + 3 \le d_{C_n}(v', v'') \le 2a_k - 2$. Then $3 \le d_{C_n}(v''', v') \le a_k - 2$ and we have $d_{C_n}(v''', v') - 1 \le a_k - 3$ and $a_k - d_{C_n}(v''', v') \le a_k - 3$; hence, for $0 \le t \le a_k$,

$$\begin{aligned} & d_G\left(v, v_t\right) \\ & \quad \le \left\lfloor \frac{n}{2a_k} \right\rfloor + 1 \\ & \qquad + \frac{1}{2}\max\left\{d_{C_n}\left(v''', v'\right) - 1,\ a_k - d_{C_n}\left(v''', v'\right)\right\} \\ & \quad \le \left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k - 1}{2}. \end{aligned} \quad (12)$$

Using a symmetric argument we obtain the same inequality for $a_k < t \le d_{C_n}(v', v'')$.

Hence, one can check that

$$d_G\left(v, p\right) \le \left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k - 1}{2} \quad (13)$$

holds for every $p \in G$, if $0 \le d_{C_n}(v', v'') \le a_k - 1$ or $a_k + 3 \le d_{C_n}(v', v'') \le 2a_k - 2$. Since G is equal to the closed ball or radius $\lfloor n/(2a_k) \rfloor + (a_k - 1)/2$ and center v for every $v \in V(G)$, we conclude $\operatorname{diam} G \le \lfloor n/(2a_k) \rfloor + a_k/2$ in this case.

Assume now that $d_{C_n}(v',v'') \in \{a_k+2, 2a_k-1\}$. Then $d_{C_n}(v''',v') \in \{2, a_k-1\}$ and we have $d_{C_n}(v''',v')-1 \le a_k-2$ and $a_k - d_{C_n}(v''',v') \le a_k-2$; hence, for $0 \le t \le a_k$,

$$\begin{aligned} d_G(v,v_t) &\le \left\lfloor \frac{n}{2a_k} \right\rfloor + 1 \\ &\quad + \frac{1}{2}\max\left\{d_{C_n}\left(v''',v'\right)-1, a_k - d_{C_n}\left(v''',v'\right)\right\} \\ &\le \left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k}{2}. \end{aligned} \tag{14}$$

Using a symmetric argument we obtain the same inequality for $a_k < t \le d_{C_n}(v',v'')$. Hence,

$$d_G(v,p) \le \left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k}{2} \tag{15}$$

holds for every $p \in G$, and we conclude $\operatorname{diam} G \le \lfloor n/(2a_k)\rfloor + (a_k+1)/2$ in this case.

If $d_{C_n}(v',v'') = a_k$ (n is an odd multiple of a_k), then a similar argument gives $d_G(v,p) \le \lfloor n/(2a_k)\rfloor + a_k/2$ for every $p \in G$. If z is the midpoint of $[u,v] \in E(G)$, then the previous argument gives

$$\begin{aligned} \overline{B\left(z, \left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k}{2}\right)} &= \overline{B\left(u, \left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k-1}{2}\right)} \\ &\quad \cup \overline{B\left(v, \left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k-1}{2}\right)} \\ &= G. \end{aligned} \tag{16}$$

Hence, we also obtain $\operatorname{diam} G \le \lfloor n/(2a_k)\rfloor + a_k/2$ in this case.

If $d_{C_n}(v',v'') = a_k+1$, then a similar argument gives $d_G(v,p) \le \lfloor n/(2a_k)\rfloor + (a_k+1)/2$ for every $p \in G$. If z is the midpoint of $[u,v] \in E(G)$, then the previous argument gives

$$\begin{aligned} &\overline{B\left(z, \left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k+1}{2}\right)} \\ &\quad = \overline{B\left(u, \left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k}{2}\right)} \cup \overline{B\left(v, \left\lfloor \frac{n}{2a_k} \right\rfloor + \frac{a_k}{2}\right)} = G. \end{aligned} \tag{17}$$

Thus, we obtain $\operatorname{diam} G \le \lfloor n/(2a_k)\rfloor + (a_k+1)/2$ in this case.

Therefore, Theorem 9 gives the desired inequalities.

Assume now that $k=2$ and n is an odd multiple of a_2. Define $t := \lfloor a_2/2\rfloor$ and $s := \lfloor n/(2a_2)\rfloor$. Fix a vertex $v \in V(G)$, and denote by $v_1,\dots,v_t,v_1',\dots,v_t',z_1,\dots,z_s,z_1',\dots,z_s'$ vertices with $d_{C_n}(v,v_j)=j$ for $1 \le j \le t$, $d_{C_n}(v_j,v_{j+1})=1$ for $1 \le j < t$, $d_{C_n}(z_j,v) = t + ja_2 = t + d_{C_n}(z_j,v_t)$ for $1 \le j \le s$, $d_{C_n}(z_j',v) = ja_2$ for $1 \le j \le s$, $d_{C_n}(z_j',z_{j+1}') = a_2$ for $1 \le j < s$, $d_{C_n}(v_j',v) = sa_2 + j = sa_2 + d_{C_n}(v_j',z_s')$ for $1 \le j \le t$, and $d_{C_n}(z_1',v_1) = 1 + a_2$. Define

$$\begin{aligned} \gamma_0 &:= [v,v_1] \cup [v_1,v_2] \cup \cdots \cup [v_{t-1},v_t] \cup [v_t,z_1] \\ &\quad \cup [z_1,z_2] \cup \cdots \cup [z_{s-1},z_s], \\ \gamma_0' &:= \left[v,z_1'\right] \cup \left[z_1',z_2'\right] \cup \cdots \cup \left[z_{s-1}',z_s'\right] \cup \left[z_s',v_1'\right] \\ &\quad \cup \left[v_1',v_2'\right] \cup \cdots \cup \left[v_{t-1}',v_t'\right]. \end{aligned} \tag{18}$$

Since n is an odd multiple of a_2, we have

$$\begin{aligned} d_{C_n}(v,z_s) &= d_{C_n}\left(v,v_t'\right) = t + sa_2 = \left\lfloor \frac{n}{2a_2} \right\rfloor a_2 + \left\lfloor \frac{a_2}{2} \right\rfloor \\ &= \frac{n}{2} + \left\lfloor \frac{a_2}{2} \right\rfloor - \frac{a_2}{2}. \end{aligned} \tag{19}$$

Hence, $z_s = v_t'$ if a_2 is even and $d_{C_n}(z_s,v_t') = 1$ if a_2 is odd; let w be the midpoint of $[z_s v_t']$ and define $\gamma := \gamma_0 \cup [z_s w]$ and $\gamma' := \gamma_0' \cup [v_t' w]$. Then γ and γ' are geodesics and $L(\gamma) = L(\gamma') = d_G(v,w) = \lfloor n/(2a_2)\rfloor + a_2/2$. Let T be the geodesic bigon $T = \{\gamma,\gamma'\}$ and p the midpoint of γ. We have

$$\begin{aligned} \delta(G) &\ge d_G\left(p,\gamma'\right) = d_G\left(p,\{v,w\}\right) = \frac{1}{2} d_G(v,w) \\ &= \frac{1}{2}\left\lfloor \frac{n}{2a_2} \right\rfloor + \frac{a_2}{4}. \end{aligned} \tag{20}$$

Since we have proved the converse inequality, we conclude that the equality holds.

Assume that $k=2$, a_2 is odd and $n - 2a_2\lfloor n/(2a_2)\rfloor = a_2+1$. We obtain the equality by using a similar bigon to the previous case, with $t := (a_2+1)/2$.

Finally, we prove the lower bound. By Theorem 11, we can assume that $\lfloor n/(2a_k)\rfloor \ge 2$.

Let us define $x = 0$, $y = a_k\lfloor n/(2a_k)\rfloor$, and $z = n - a_k\lfloor n/(2a_k)\rfloor$ (if n is a multiple of $2a_k$, then $y = z$). Consider the geodesics

$$\begin{aligned} &[xy] \\ &\quad := [0,a_k] \cup [a_k,2a_k] \cup \cdots \\ &\qquad \cup \left[\left(\left\lfloor \frac{n}{(2a_k)} \right\rfloor - 1\right)a_k, \left\lfloor \frac{n}{(2a_k)} \right\rfloor a_k\right], \\ &[xz] \\ &\quad := [0,n-a_k] \cup [n-a_k,n-2a_k] \cup \cdots \\ &\qquad \cup \left[n - \left(\left\lfloor \frac{n}{(2a_k)} \right\rfloor - 1\right)a_k, n - \left\lfloor \frac{n}{(2a_k)} \right\rfloor a_k\right]. \end{aligned} \tag{21}$$

We define an appropriate geodesic $[yz]$ in the following way. A geodesic g can be obtained as

$$\begin{aligned} g &:= \left[y, y + a_{j_1}\right] \cup \left[y + a_{j_1}, y + a_{j_2}\right] \cup \cdots \\ &\cup \left[y + \sum_{i=1}^{r-1} a_{j_i}, y + \sum_{i=1}^{r} a_{j_i}\right] \\ &\cup \left[y + \sum_{i=1}^{r} a_{j_i}, y + \sum_{i=1}^{r} a_{j_i} - a_{j'_1}\right] \\ &\cup \left[y + \sum_{i=1}^{r} a_{j_i} - a_{j'_1}, y + \sum_{i=1}^{r} a_{j_i} - a_{j'_1} - a_{j'_2}\right] \cup \cdots \\ &\cup \left[y + \sum_{i=1}^{r} a_{j_i} - \sum_{i=1}^{r'-1} a_{j'_i}, y + \sum_{i=1}^{r} a_{j_i} - \sum_{i=1}^{r'} a_{j'_i}\right], \end{aligned} \tag{22}$$

with $r \geq r' \geq 0$ and $r + r' = d_G(y, z)$ (if $r' = 0$, then the part of g with negative numbers does not appear).

Let us define the finite sequence $\{t_1, \ldots, t_r\}$ in the following way:

$$t_s := \max\left\{m \in \mathbb{N} \mid \sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{m} a_{j'_i} \geq 0\right\} \tag{23}$$

if $\sum_{i=1}^{s} a_{j_i} - a_{j'_1} \geq 0$, and $t_s := 0$ otherwise (i.e., we define $\sum_{i=1}^{0} a_{j'_i} = 0$). It is clear that $t_1 \leq \cdots \leq t_r$ and $t_r = r'$.

Consider $1 < s \leq r$. If $t_{s-1} = 0$, then

$$\begin{aligned} &\sum_{i=1}^{s-1} a_{j_i} < a_{j'_1}, \\ &\sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{t_{s-1}} a_{j'_i} = \sum_{i=1}^{s-1} a_{j_i} + a_{j_s} < a_{j'_1} + a_{j_s} \leq 2a_k. \end{aligned} \tag{24}$$

If $t_{s-1} = r'$, then

$$\begin{aligned} \sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{t_{s-1}} a_{j'_i} &= \sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{r'} a_{j'_i} \leq \sum_{i=1}^{r} a_{j_i} - \sum_{i=1}^{r'} a_{j'_i} = z - y \\ &< 2a_k. \end{aligned} \tag{25}$$

If $0 < t_{s-1} < r'$, then

$$\sum_{i=1}^{s-1} a_{j_i} - \sum_{i=1}^{t_{s-1}} a_{j'_i} < a_{j'_{t_{s-1}+1}} \leq a_k, \tag{26}$$

and so,

$$\sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{t_{s-1}} a_{j'_i} = a_{j_s} + \sum_{i=1}^{s-1} a_{j_i} - \sum_{i=1}^{t_{s-1}} a_{j'_i} < a_{j_s} + a_k \leq 2a_k. \tag{27}$$

Let us define

$$\begin{aligned} \gamma_1 &:= \left[y, y + a_{j_1}\right] \cup \left[y + a_{j_1}, y + a_{j_1} - a_{j'_1}\right] \\ &\cup \left[y + a_{j_1} - a_{j'_1}, y + a_{j_1} - a_{j'_1} - a_{j'_2}\right] \cup \cdots \\ &\cup \left[y + a_{j_1} - \sum_{i=1}^{t_1-1} a_{j'_i}, y + a_{j_1} - \sum_{i=1}^{t_1} a_{j'_i}\right], \\ \gamma_s &:= \left[y + \sum_{i=1}^{s-1} a_{j_i} - \sum_{i=1}^{t_{s-1}} a_{j'_i}, y + \sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{t_{s-1}} a_{j'_i}\right] \\ &\cup \left[y + \sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{t_{s-1}} a_{j'_i}, y + \sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{t_{s-1}+1} a_{j'_i}\right] \\ &\cup \cdots \\ &\cup \left[y + \sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{t_s-1} a_{j'_i}, y + \sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{t_s} a_{j'_i}\right], \end{aligned} \tag{28}$$

for $1 < s \leq r$ (if $t_s = t_{s-1}$, then γ_s is a single edge). We have

$$\begin{aligned} L(\gamma_1) &= 1 + t_1, \\ L(\gamma_s) &= 1 + t_s - t_{s-1}, \quad 1 < s \leq r, \\ L(\gamma_1 \cup \cdots \cup \gamma_r) &= r + t_r = r + r' = d_G(y, z). \end{aligned} \tag{29}$$

Since $\gamma_1 \cup \cdots \cup \gamma_r$ joins y and z, and $L(\gamma_1 \cup \cdots \cup \gamma_r) = d_G(y, z)$, we consider the geodesic $[yz] := \gamma_1 \cup \cdots \cup \gamma_r$.

Since

$$\sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{t_s} a_{j'_i} \geq 0, \tag{30}$$

we have $v \geq y$ for every $v \in [yz] \cap V(G)$.

Since $\lfloor n/(2a_k) \rfloor \geq 2$, we have

$$\sum_{i=1}^{s} a_{j_i} - \sum_{i=1}^{t_{s-1}} a_{j'_i} \leq 2a_k \leq n - z \leq n - y, \tag{31}$$

and we conclude that $y \leq v \leq n$ for every $v \in [yz] \cap V(G)$. Let p be the midpoint of $[xy]$. Since $y \leq v \leq n$ for every $v \in ([yz] \cap [xz]) \cap V(G)$,

$$\begin{aligned} \delta(G) &\geq d_G(p, [yz] \cup [xz]) = d_G(p, \{x, y\}) \\ &= \frac{1}{2} L([xy]) = \frac{1}{2} \left\lfloor \frac{n}{2a_k} \right\rfloor. \end{aligned} \tag{32}$$

□

The lower bound in Theorem 28 can be improved for $k = 2$.

Theorem 29. *For any integers $1 < a_2 \leq \lfloor n/2 \rfloor$ with $n - 2a_2 \lfloor n/(2a_2) \rfloor \leq a_2 + 1$ one has*

$$\delta(C_n(1, a_2)) \geq \frac{1}{2} \left\lfloor \frac{n}{2a_2} \right\rfloor + \frac{1}{4} \left(n - 2a_2 \left\lfloor \frac{n}{2a_2} \right\rfloor\right). \tag{33}$$

The equality in (33) is attained if $n - 2a_2 \lfloor n/(2a_2) \rfloor = a_2 + 1$.

Proof. Let us denote by $\{0, 1, \ldots, n-1\}$ the vertices of $G := C_n(1, a_2, \ldots, a_k)$, and let us denote by C_n the subgraph of G with $V(C_n) = V(G)$ and $E(C_n) = \{[0,1], [1,2], \ldots, [n-2, n-1], [n-1, 1]\}$.

Let $v = 0$, $v' = a_2\lfloor n/(2a_2)\rfloor$, and $v'' = n - a_2\lfloor n/(2a_2)\rfloor$. Thus $d_{C_n}(v, v') = d_{C_n}(v, v'') = a_2\lfloor n/(2a_2)\rfloor$ (if n is a multiple of $2a_2$, then $v' = v''$); therefore, $d_{C_n}(v', v'') = n - 2a_2\lfloor n/(2a_2)\rfloor$ and $d_G(v, v') = d_G(v, v'') = \lfloor n/(2a_2)\rfloor$. Note $0 \le d_{C_n}(v', v'') = n - 2a_2\lfloor n/(2a_2)\rfloor \le a_2 + 1$.

If $d_{C_n}(v', v'') = 0$, then Theorem 28 gives (33).

Assume that $0 < d_{C_n}(v', v'') \le a_2 + 1$. Define v_0 as the point in C_n with $d_{C_n}(v_0, v') = d_{C_n}(v_0, v') = d_{C_n}(v', v'')/2$. One can check that $d_G(v_0, v') = d_{C_n}(v_0, v') = d_{C_n}(v', v'')/2$. Define $g_1 := [vv']$ as the geodesic in G with $g_1 \cap V(G) = \{0, a_2, 2a_2, \ldots, a_2\lfloor n/(2a_2)\rfloor\}$ and $g_2 := [vv'']$ the geodesic in G with $g_2 \cap V(G) = \{0, n - a_2, n - 2a_2, \ldots, n - a_2\lfloor n/(2a_2)\rfloor\}$. Let $g_1' := [v'v_0]$ and $g_2' := [v''v_0]$ be geodesics in G contained in C_n. Thus $\gamma_1 := g_1 \cup g_1'$ and $\gamma_2 := g_2 \cup g_2'$ are two geodesics in G joining v and v_0. If p is the midpoint of γ_1, then

$$\begin{aligned}\delta(G) &\ge d_G(p, \gamma_2) = d_G(p, \{v, v_0\}) = \frac{1}{2}L(\gamma_1) \\ &= \frac{1}{2}\left\lfloor \frac{n}{2a_2} \right\rfloor + \frac{1}{4} d_{C_n}(v', v'') \\ &= \frac{1}{2}\left\lfloor \frac{n}{2a_2} \right\rfloor + \frac{1}{4}\left(n - 2a_2\left\lfloor \frac{n}{2a_2} \right\rfloor\right).\end{aligned} \tag{34}$$

Theorem 28 gives that the equality in (33) is attained if $n - 2a_2\lfloor n/(2a_2)\rfloor = a_2 + 1$. □

The upper bounds in Theorem 28 can be improved for k large enough.

Theorem 30. *Consider any integers $k > 1$ and $1 < a_2 < \cdots < a_k \le \lfloor n/2 \rfloor$. Assume that $k \ge (n-1)/4$ if $a_k \ne n/2$ and $k \ge (n+1)/4$ if $a_k = n/2$. Then one has*

$$1 \le \delta(C_n(1, a_2, \ldots, a_k)) \le \frac{3}{2}. \tag{35}$$

Proof. By Theorem 11, it suffices to prove the upper bound. Recall that if $a_k \ne n/2$, then $G := C_n(1, a_2, \ldots, a_k)$ is a regular graph with degree $\Delta = 2k$ and that if $a_k = n/2$, then G is regular with degree $\Delta = 2k - 1$. In any case we have $\deg(v_1) + \deg(v_2) = 2\Delta \ge n - 1$ for every $v_1, v_2 \in V(G)$. Therefore, given any $u, v \in V(G)$ with $d_G(u, v) > 1$, there exists a vertex w with $d_G(u, w) = d_G(v, w) = 1$; thus $d_G(u, v) = 2$ and we conclude that $\operatorname{diam} V(G) = 2$. Then Corollary 10 gives $\delta(G) \le 3/2$. □

Note that by Theorem 12, under the hypothesis in Theorem 30, the possible values for $\delta(C_n(1, a_2, \ldots, a_k))$ are just 1, 5/4, and 3/2.

Finally, the next result estimates the hyperbolicity constant if $a_k = k$.

Theorem 31. *For any integers $1 < k \le \lfloor n/2 \rfloor$ one has*

$$\begin{aligned}&\frac{1}{2}\left\lfloor \frac{n}{2k} \right\rfloor + \frac{1}{2}\left(\frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor\right) \\ &\quad \le \delta(C_n(1, 2, 3, \ldots, k-1, k)) \le \frac{1}{2}\left\lfloor \frac{n}{2k} \right\rfloor + \frac{1}{2},\end{aligned} \tag{36}$$

if $n = 0, 1 (\operatorname{mod} 2k)$, and

$$\begin{aligned}&\frac{1}{2}\left\lfloor \frac{n}{2k} \right\rfloor + \frac{1}{2} + \frac{1}{2}\left(\frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor\right) \\ &\quad \le \delta(C_n(1, 2, 3, \ldots, k-1, k)) \le \frac{1}{2}\left\lfloor \frac{n}{2k} \right\rfloor + 1,\end{aligned} \tag{37}$$

if $n \ne 0, 1 (\operatorname{mod} 2k)$.

Proof. Let us denote by $\{0, 1, \ldots, n-1\}$ the vertices of $G := C_n(1, 2, 3, \ldots, k-1, k)$.

It is easy to check that

$$\begin{aligned}\operatorname{diam} V(G) &= \left\lfloor \frac{n}{2k} \right\rfloor \quad \text{if } n = 0, 1 (\operatorname{mod} 2k), \\ \operatorname{diam} V(G) &= \left\lfloor \frac{n}{2k} \right\rfloor + 1 \quad \text{if } n \ne 0, 1 (\operatorname{mod} 2k),\end{aligned} \tag{38}$$

and Corollary 10 gives the upper bounds.

In order to prove the lower bounds, assume first that n is even (and then $n/2 - \lfloor n/2 \rfloor = 0$).

Define g_1, g_2 as the geodesics in G joining 0 and $n/2$ with $g_1 \cap V(G) = \{0, k, 2k, \ldots, k\lfloor n/(2k)\rfloor, n/2\}$ and $g_2 \cap V(G) = \{0, n-k, n-2k, \ldots, n - k\lfloor n/(2k)\rfloor, n/2\}$. If p is the midpoint of g_1, then

$$\begin{aligned}\delta(G) &\ge d_G(p, g_2) = d_G\left(p, \left\{0, \frac{n}{2}\right\}\right) = \frac{1}{2}L(g_1) \\ &= \frac{1}{2}\operatorname{diam} V(G).\end{aligned} \tag{39}$$

Assume now that n is odd (and then $n/2 - \lfloor n/2 \rfloor = 1/2$) and let y be the midpoint of the edge $[(n-1)/2, (n+1)/2]$.

Define g_1', g_2' as the geodesics in G joining 0 and y with $g_1' \cap V(G) = \{0, k, 2k, \ldots, k\lfloor n/(2k)\rfloor, (n-1)/2\}$ and $g_2' \cap V(G) = \{0, n-k, n-2k, \ldots, n - k\lfloor n/(2k)\rfloor, (n+1)/2\}$. If p' is the midpoint of g_1', then

$$\begin{aligned}\delta(G) &\ge d_G(p', g_2') = d_G\left(p, \left\{0, \frac{n}{2}\right\}\right) = \frac{1}{2}L(g_1') \\ &= \frac{1}{2}\left(\operatorname{diam} V(G) + \frac{1}{2}\right).\end{aligned} \tag{40}$$

□

4. Conclusions

In this paper we study the hyperbolicity constant of an important class of networks: circulant graphs. We obtain several sharp inequalities for the hyperbolicity constant and in some cases we characterize the graphs for which the equality is attained.

Theorem 3 in Section 2 gives the precise value of the hyperbolicity constant of $\delta(C_n(a_1))$. Theorem 11 provides a sharp lower bound for $\delta(C_n(a_1, a_2, \ldots, a_k))$ and characterizes the graphs for which the equality is attained. It is well known that a network is circulant if and only if its complement is circulant. Thus it is natural to study in this context the properties of general complement graphs. In Theorems 15 and 24 this kind of results for general networks appears and they are applied to circulant graphs in Corollary 25.

We collect in Section 3 several sharp inequalities for the hyperbolicity constant of a large class of circulant graphs. In Theorem 28 good lower and upper bounds for $\delta(C_n(1, a_2, \ldots, a_k))$ appear, which are improved in Theorems 29, 30, and 31 with additional hypothesis. Furthermore, we obtain the precise value of the hyperbolicity constant of many circulant networks (see Theorems 3, 11, and 29 and Corollary 25).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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