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Exploring mathematical connections of pre-university students through tasks involving rates of change

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ABSTRACT
This paper reports the results of a research exploring the mathematical connections of pre-university students while they solving tasks which involving rates of change. We assume mathematical connections as a cognitive process through which a person finds real relationships between two or more ideas, concepts, definitions, theorems, procedures, representations or meanings or with other disciplines or the real-world. Four tasks were proposed to the 33 pre-university students that participated in this research; the central concept of the first task is the slope, the last three tasks contain concepts like velocity, speed and acceleration. Task-based interviews were conducted to collect data and later analysed with thematic analysis. Results showed most of the students made mathematical connections of the procedural type, the mathematical connections of the common features type are made in smaller quantities and the mathematical connection of the generalization type is scarcely made. Furthermore, students considered slope as a concept disconnected from velocity, speed and acceleration.

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KEYWORDS
Mathematical connections; rate of change; slope; velocity; speed; acceleration; pre-university students

1. Introduction

Mexican curricula suggest studying the rate of change associated with the slope and inclination of a straight line from ninth grade[^1] [1]. The rate of change is associated with ratio, proportion and proportional variation in tenth grade [2]. The algebraic construction of the slope and its implications with parallelism and perpendicularity is studied in eleventh grade [3]. The instant rate of change associated with the slope, velocity, speed and acceleration is used to construct the concept of the derivative in the twelfth grade [4]. The previous concepts are also studied in the Physics course in the eleventh grade [5]. In addition, the study of the American Mathematical Curriculum Standards of Stanton and Moore-Russo [6] showed that this concept is introduced in eighth grade. The slope is considered as a geometrical ratio or constant rate of change in most of the states in the United States of America, and third of the 50 states associates the slope to real-world situations, involving physical and functional situations.

The rate of change is an important concept in mathematical education for several reasons: it is a ‘powerful linking concept’ [7,p.54]) for understanding functions and their

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graphs; it is fundamental for those who consider further university studies to understand functions [8] and it contributes to the understanding of the concept of slope through real-life situations [9]. The slope of a linear function represents the rate of change in one variable that results from a change in the other variable, so it plays an important role in understanding issues involving to the rate of change. However, several researchers have documented the difficulties and misconceptions that students often have when they dealing with the concept of slope. One such example being the confusion that often arises regarding height and slope, due to the fact that students belief that a greater slope tends to be associated with a greater height or vertical distance [7,10].

Other difficulties are identified in the interpretation of linear functions, in particular, concerning the connection of the graph with its equation and the rate of change [7,11]. According to Birgin [11], this is mostly due to confusion regarding the role of $m$ and $b$ when the linear function is presented in the form $y = mx + b$, as the students appear unable to distinguish between the values of the intersections in the $x$-axis and $y$-axis. In the same way, considering the slope as a rate of change leads to interpretation difficulties when $m$ is an integer. In line with Deniz and Kabael [12], this is because even though the students accept the slope as the rise over run change, they regard it as an algorithm, not as an interpretation of changes. Similarly, Cho and Nagle [13] reported that university students memorize the rise over run, do not express it in its simplest form when they write it in fraction form, besides not coordinating the amount of change in the dependent variable by the amount of change in the independent variable. Furthermore, when they graph a straight line they do it in the opposite direction to the given slope. Hence, some students understand the concept of the slope as a parametric coefficient $m$, but limited to a very procedural interpretation which does not allow understand it as an indicator of the rate of change of two covalent variables or in its graphic representation [12,13].

Another aspect that Zaslavsky, Sela and Leron [14] identified in the students (including teachers) is that for the description of the slope of two lines presented in non-homogeneous systems, the visual perspective is more important than the analytical one, so they interpret the slope by the angle, as also reported by Stump [9]. However, Zaslavsky and colleagues [14] consider that in order to improve the understanding of the slope, a distinction must be made between the visual slope (slope of a straight line) and the analytical slope (rate of change of a function) in which discussions are embedded in the context of the non-homogeneous scale change. On the other hand, Stump [9] found that the slope as a rate of change is acceptable by students and teachers, but does not rule out the difficulty that they have when attempting to interpret it.

Despite the relevance of the rate of change in the curricula and mathematical research, pre-university [9,15–17] and university students [13,18–21] still have trouble understanding this concept. It is a difficult concept even for teachers [14,22]. In this sense, Herbert and Pierce [23] reported that students are able to calculate the slopes numerically but with a little approach to the interpretation of meanings in a specific context and without considering the measuring unit. In the same way, the investigation of Stump [9] focused on the understanding of the slope in real-world situations found that students have a better understanding of the slope in functional situations but still have problems with its interpretation. As a consequence, it is suggested that instruction allow students to make mathematical connections between different mathematical domains and real-life situations [24–26] and provide opportunities to improve mathematical understanding. In this regard,
NCTM [27] proposes that students must be able to recognize and use connections between mathematical ideas and use them in different contexts. The research community in mathematical education has given little attention to the rate of change despite its importance in pre-university and university curricula, in addition to the problems and difficulties in its understanding. In fact, little is known about the mathematical connections between the rate of change and its different features like slope, velocity, speed and acceleration. Therefore, we look for answers to the following research question in this work: Which mathematical connections do students make while engaging in tasks involving rates of change like slope, velocity, speed and acceleration? The pre-university level includes grades 10th, 11th and 12th grade; this level is equivalent to High School in the U.S.A.

2. Theoretical framework

2.1. Rate of change

Stewart [28] defines the average rate of change of $y$ with respect to $x$ as the quotient of their differences: $\Delta y/\Delta x = (f(x_2) - f(x_1))/(x_2 - x_1)$ in the interval $[x_1, x_2]$. This expression is interpreted in graphical terms as the slope of the secant line of the curve $f$. However, the measure of change in real-world situations is the most meaningful interpretation. Real-world representations of slope exist in two different forms: physical situations, such as mountain pathways, ski slope and wheelchair ramps; and functional situations, such as distance versus time or quantity versus costs [9]. The most elemental change in real-world situations happens when a changing magnitude increases or decreases in a constant way, this quantity is calculated by the quotient of the differences of $\Delta y$ and $\Delta x$, so the expression $\Delta y/\Delta x$ is a ratio or rate between changes. According to Stump [9], in both physical situations and functional situations, the slope can be thought of as either a ratio or a rate, depending on the level of reflective abstraction. The measuring units highlight the idea that the rate connects with the real-world situation [6]; we consider that they give context and specificity to the rates of change: m/s in the case of velocity and m/s$^2$ in the case of acceleration.

2.2. Mathematical connections

The mathematical connections involve establishing relationships between different mathematical concepts [29,30]. They are networks of links that coordinate definitions, properties, techniques and procedures to build inter-concepts. These links are logical and coherent links between representations [31]. In a cognitive sense, Eli, Mohr-Schroeder and Lee [32] assumed that mathematical connections can be described as components of a scheme or connected groups of schemes in a mental network. They also consider, from a constructivist approach, that a mathematical connection can be seen as a link (or bridge) that uses previous or new knowledge to establish or strengthen the understanding of the relationship(s) between two or more mathematical ideas, concepts, strands or representations in a mental network.

On the other hand, Singletary [33] presents three interpretations of the mathematical connections: as a fundamental characteristic of mathematics, that is to say as an inherent
part of the discipline; as the product of the understanding; and as part of the process of doing mathematics. Businskas [34] has already pointed out these interpretations and highlighted that the thinking processes use these connections to build mathematics. Evitts [35] is consistent with this idea by considering that the connected knowledge can be described in terms of its personal construction and meaning, the multiple links between concepts and procedures and the power that comes from knowing these connections.

There can also be connections with the real-world, previous knowledge, familiar contexts (inside and outside the school), other disciplines as well as future and past [36–38]. Therefore, the mathematical topics can be linked to them (intra-mathematical connections), to topics of other disciplines or contexts, specifically solving application problems (extra-mathematical connections) [39]. The common characteristic of all these different definitions or classifications of mathematical connections is that they are the bonds or bridges between the mathematical ideas. For the purpose of this investigation, we assume mathematical connections as a cognitive process through which a person finds true relationships between two or more ideas, concepts, definitions, theorems, procedures, representations, meanings between them, or between these with other disciplines or real life [39]. Mathematical connections emerge when students carry out specific tasks; they can be identified in their written productions or in the arguments produced while carrying out these tasks.

3. Method

This is a qualitative research that used individual task-based interviews to collect data. According to Goldin [40], the task-based interviews to study mathematical behaviour, involve a minimum interaction between a subject (the problem solver) and an interviewer (the clinician) engaged in one or more tasks (questions, problems or activities) introduced by the clinician in a previously planned strategy. Goldin claims that the researcher can make inferences on the mathematical thinking, learning or problem solving from the analysis of the behaviour of verbal or non-verbal interactions. In this regard, Assad [41] highlights that task-based interviews not only give opportunities to evaluate the conceptual knowledge of the students but widen this comprehension. According to Assad, the protocol of the interview may be structured with guidelines and previously planned answers by the interviewer, it can also be introduced as semi-structured interviews that allow the interviewer to judge the adequate response to the mathematical reasoning of the student.

3.1. Design of task-based interviews

The protocol of the task-based interviews included four tasks. The first one has an intra-mathematical character and asks for the slope of a secant line. The other three tasks refer to functional situations linked to the real world: velocity, speed and acceleration associated with the real-world (see Appendix). The construction of these tasks is based on our interest to explore the relationships between the slope and the functional situations previously mentioned and because these tasks are suggested in the current mathematics [4] and physics [5] curricula of Mexico. These programmes suggest associating the slope and the rate of change with natural, social, economic and administrative phenomena through the concepts of velocity, speed and other special cases of the rate of change. Therefore, we were
expecting students that were familiarized with these tasks and its implementation would bring out the desired connections. Before its final implementation, the tasks and interviews were tested twice with four pre-university students, unrelated to the participants, to prove accessibility and efficacy and to understand how it works, the type of results produced and to refine this tool in function of the goal.

The researchers asked some questions while the students dealt with the tasks. The students were asked to clarify how they solved the task and obtained the data if this was not mentioned explicitly. Another question focused on the meaning of the result obtained. At the end of the four tasks, they were asked if they noted some relationships between the tasks and what those relationships were. If they did not identify any relationship then they were asked about the relationships between the formulas used, their similitudes and differences?

The interviews were conducted at the beginning of the first week of the training course for university offered in August 2017; the interviews were individual, videotaped and transcribed for their analysis.

3.2. Participants

The 33 students participants in this study, 19 men and 14 women between ages 17 and 20 years old, have recently finished their pre-university studies. They all come from different regions of the state of Guerrero in Mexico. The minimum grade obtained in their mathematical courses was 8, in a scale from 0 to 10, so they were all considered as university applicants for mathematics. They were all registered in a training course for the university; this course aims to fill the gaps of their mathematical education to begin their university studies in Mathematics and is focused on the topics of Arithmetic, Algebra, Geometry and Calculus. The students attended the course for 8 hours a day for four weeks.

3.3. Data analysis

We used the thematic analysis suggested by Braun and Clarke [42,43] to analyse the data and the triangulation method so as to provide reliability, validity, credibility and rigour [44]. The aim of the thematic analysis is to identify patterns of meanings (themes) using a set of data to give answers to the research question. According to Braun and Clarke [42], a theme captures something important of the data relative to the research question and represents some level of answer or modelled meaning inside the group of data. These patterns are identified through a rigorous process of familiarization and codification of data and development and review of themes. Thematic analysis can be used with a wide range of theoretical frameworks and even for several research questions. It is useful to analyse different types of data, this means that it allows the work of big and small sets of data. Finally, it may produce analysis based on the data or conducted by a specific theory. This method is structured in the following phases: Familiarizing yourself with the data; generating initial codes; searching for themes; reviewing the themes; defining and naming the themes; and producing the report.

Triangulation of data is a heuristic procedure designed to document information according to different points of view [45]. For Aguilar and Barroso [44], triangulation implies having several observers in the field, thus increasing the quality and validity of the
data, since the bias of a single researcher is thus eliminated. Consequently, for this study, the researchers (all with different levels of experience) analysed the data during the first 4 phases independently, subsequently, they compared and discussed their results. In case of disagreement, the data would be analysed jointly in and this process would eventually lead to a consensus of opinion. All this was done in phase 4 of the thematic analysis. In phase 5 the three authors discussed the results and formed conclusions in specific work sessions. Next, we describe each of the phases from which the method is structured.

Phase 1. *Familiarizing yourself with the data.* We became familiar with the data and the language used by the participants by repeatedly reading the transcriptions of the interviews.

Phase 2. *Generating initial codes.* We established initial codes for a first classification based on the previous lectures of the narratives. In this phase, we identified words or phrases indicating mathematical connections between ideas, concepts, procedures or meanings. Accordingly, we uncovered phrases such as ‘the slope is calculated with . . . ’, ‘I used the graph to . . . ’, ‘the formulas are similar or different because . . . ’, ‘the meaning of the result is . . . ’, ‘I remember this is the formula for . . . ’, etc. For example, one can see from the sentences in italics from the excerpt below that a number of codes were established: the graph provides data, the formula of the slope is \((y_2 - y_1)/(x_2 - x_1)\) and, the slope is associated with \(m\).

Saraí: *The data provided by the graph are, point A is located in (1,1) since this quadrant is positive, in point B we have x is 3, and the point is located in 3 to 9, this is the data that we have, well, and the formula of the slope \([\text{write } m] \) is equal to \(y\) two minus \(y\) one divided by \(x\) two minus \(x\) one \([\text{write } m = (y_2 - y_1)/(x_2 - x_1)]\).*

Phase 3. *Searching for themes.* We compared the excerpts associated with each of the initial codes looking for themes. This allowed us to cluster the associated patterns of response of the students to the tasks into themes. In terms of this research, each theme is a mathematical connection. For example, from the codes: the formula to calculate the average speed is \((h_2 - h_1)/(t_2 - t_1)\), the formula of the slope is \((y_2 - y_1)/(x_2 - x_1)\) and the formula of the acceleration is final velocity minus initial velocity divided by final time minus the initial time, the theme was constructed: the terms are first associated with a formula.

Phase 4. *Reviewing the themes.* The correspondence, among the themes identified in the previous phase and the data, was made. As a result, some themes were modified and those that did not have enough evidence to support students’ ideas were eliminated. The work sessions held by the three authors allowed us to reach consensus regarding the final themes identified in this study and these are discussed in the results section.

Phase 5. *Defining and naming the themes.* The themes were defined, named and clustered into three categories corresponding with the categories of the mathematical connections found: use of different representations, procedural and common features (see Table 1). We identified the relationships among the connections made by the students while solving the tasks.

Phase 6. *Producing the report.* The report includes the themes recently defined and clustered into categories containing the mathematical connections found.
Table 1. Mathematical connections identified in the solved tasks.

<table>
<thead>
<tr>
<th>Categories</th>
<th>Mathematical connections</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Use of different representations</td>
<td>The terms ‘slope’, ‘velocity’, ‘acceleration’ and ‘speed’ are related to a formula.</td>
<td>45</td>
</tr>
<tr>
<td></td>
<td>The graph gives the coordinates of the points.</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>The term ‘slope’ is associated with the letter m.</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>The result as a change of y with respect to the change of x.</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>We can estimate the differences through the graphs.</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>s(t) gives heights or distances.</td>
<td>1</td>
</tr>
<tr>
<td>Procedurals</td>
<td>The slope is found with the formula ( \frac{y_2 - y_1}{x_2 - x_1} ).</td>
<td>22</td>
</tr>
<tr>
<td></td>
<td>The acceleration is found with the formula ( a = \frac{v_2 - v_1}{t_2 - t_1} ).</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>The speed is found with the formula ( \frac{s_2 - s_1}{t_2 - t_1} ) or ( R = \frac{P_F - P_i}{t_f - t_i} ) or ( R = \frac{NP}{T} ).</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>The velocity is found with the formula ( \frac{h_f - h_i}{t_f - t_i} ).</td>
<td>4</td>
</tr>
<tr>
<td>Common features</td>
<td>Quotients and differences are required in all formulas.</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>The formula for the speed of population growth is similar to the formula for slope.</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>The formula for slope allowed us to find the velocity, the speed of population growth and the acceleration.</td>
<td>2</td>
</tr>
</tbody>
</table>

4. Results

We identified three categories of mathematical connections while the students were dealing with the tasks (Table 1): use of different representations, procedurals and common features.

4.1. Mathematical connections associated with the use of different representations

It is natural to resort to the use of representations while working with concepts that are not directly accessible [46], like the mathematical objects. The use of these representations is the main tool to support and widen the mathematical reasoning that can be presented as numerical systems, algebraic symbolism, graphics, diagrams, models, equations, annotations, images, analogies, metaphors, histories or games [47]. This category includes mathematical connections like written, algebraic, numerical, graphical and verbal representations.

4.1.1. The terms ‘slope’, ‘velocity’, ‘acceleration’ and ‘speed’ are associated with a formula

This mathematical connection considers the relationship between the statement of the task and the formula. The students associated an algebraic representation (a formula) to the specific terms used for the rate of change in each task. In the case of task 1, 20 students associated a formula with the term ‘slope’, they did it in three ways: 11 wrote it by means of equality: \( m = \frac{(y_2 - y_1)}{(x_2 - x_1)} \); 7 students only wrote the difference quotient: \( (y_2 - y_1)/(x_2 - x_1) \), and two did the same as the first ones, only instead of using \( m \) they
used the literal $P$. Something similar occurred in the rest of the tasks: the terms ‘velocity’, ‘speed’ and ‘acceleration’ triggered the relationship with the formulas instead of using the given graphs. Four students associated the formula of the velocity in terms of the differences $v = (s_2 - s_1)/(t_2 - t_1)$ in task 2. In task 3, eleven students associated speed with formulas in terms of differences (e.g. $r = (y_2 - y_1)/(x_2 - x_1)$ or $R = (P_f - P_i)/(t_f - t_i)$) and abbreviations (e.g. $R = NP/T$). Only eleven students found a relationship between the term ‘acceleration’ and the formula $a = (v_2 - v_1)/(t_2 - t_1)$ in task 4. They showed arguments like ‘I know, by heart, the formula for slope’, ‘I believe this is the formula for velocity’, ‘I forget the formulas’.

4.1.2. **The slope is associated with the letter $m$**

Through this mathematical connection, students relate the term ‘slope’ with the symbolic representation $m$. This connection was made by 11 students, they also established the relation of equality with the algebraic expression in terms of differences, they wrote: $m = (y_2 - y_1)/(x_2 - x_1)$, and in the interview, they confirm that this is the way the slope is represented. In the same way, 2 students write the formula incorrectly but in their verbal and written expressions they make explicit that $m$ represents the slope, as shown in the following excerpt from the interview with Yamir:

Interviewer: How did you solve the task?
Yamir: Well, we know that the slope [points out the letter $m$] equals . . . well, I don’t know if my formula is right, because I don’t remember who goes up, if $x$ or $y$, but the slope [points out an $m$ with the pencil] equals $x$ one minus $x$ two over $y$ one minus $y$ two . . .

4.1.3. **The differences in the formulas can be estimated using the graph**

This mathematical connection was identified while relating the graph to the magnitude of the differences necessary to do the calculations. Three students made this connection in task 1, two students in task 2, three students in task 3 and only 2 students in task 4. This

![Figure 1](image)

**Figure 1.** Mathematical connection between the graph and the calculations of the differences.
connection was detected when the students estimated the magnitude of the differences $y_2 - y_1$ or $x_2 - x_1$ through visualization by outlining the slope or the legs of a right triangle for speed and acceleration (see Figure 1).

### 4.1.4. The graph shows the coordinates of the points

This mathematical connection was identified when the students used the graph to obtain the Cartesian coordinates and wrote them in the form $P(x, y)$ in order to do the calculations. This mathematical connection was enhanced when the students verbally claimed they performed such procedure. Some arguments are: ‘I find the coordinates from the points in the graph’, ‘The points are given in the graph’, ‘it is possible to find the height without substituting in the function’. Four students made this mathematical connection in the functional situation that asks for the velocity, eight for the speed and eleven for the acceleration. They are all mathematical connections between graphical and numerical representation.

### 4.1.5. Interpreting the result as the displacement of ‘y’ with respect displacement of ‘x’

Three students showed arguments in this sense in task 1: ‘the slope is equal to four; this means that whenever $x$ moves 1, then $y$ moves 4 units’; ‘the slope moves one this way (referring to the displacement in the $x$ axis) and then up (referring to the displacement in the $y$ axis) and then again’. We also found similar interpretations in three students in tasks 2 and 3; for example, they argued ‘the meaning of four metres per second is that the object goes four metres up from the first second to the third second’, ‘the urban population increased by 1.5 million in 10 years and the rural population increased by 0.5 million in the same period’. It is important to highlight that five students only give the numerical result without considering the units of measure in this task. Thirteen students found the correct answer in task 4; however, only two of these students give correct interpretations in terms of the acceleration: for example: ‘the bicycle rider incremented his velocity in two point five meters per second’. The rest of these 13 students only argued in the sense of ‘acceleration is 2.5 metres per second’ without considering the coherence of the units of measure; in fact, 5 students omitted the units of measure of acceleration.

### 4.1.6. $s(t)$ represents heights and distances

This mathematical connection is only identified by Erika. She started with the algebraic representation, $v = (h_f - h_i)/(t_f - t_i)$, but used another equivalent algebraic representation to obtain the velocity; he considered $h_f$ as equivalent to $s(3)$ and $h_i$ as equivalent to $s(1)$; this means that he manipulates the expression $s(t) = 8t - t^2$ to obtain the heights and the differences as shown in Figure 2.

This is verified in the verbal arguments of her answer to task 2:

Erika: The problem gives us time and height in meters. We have to find the velocity, well, according to the given formula $s$ of $t$ equals eight $t$ minus $t$ square. I used it first with $t$ equals one and we have 8 times $t$; eight times one is eight, minus one equals seven and this is the exact height. This formula [points out the expression $s(t) = 8t - t^2$] is to define height. I used it later with 3; then this 3, I substitute $t$ with three and gives me eight times 3 minus three square, it gives me fifteen. Substituting in the formula: the major height minus the minor height over the major time minus the
minor height, or what is the same as this [points out \( s(3) \)] minus this [points out \( s(1) \)]. The result is eight metres over two seconds or 4 metres per second.

4.2. **Mathematical connections of the procedural type**

In these mathematical connections, the formulas act as the starting point and the main tool to carry out the tasks; they are carried out in an algorithmic form so they are essentially sequential. The algorithm followed by the students to work with slope, velocity, speed or acceleration is: formula, substituting values, evaluate the differences and calculate the quotient, as shown in Table 2.

The written productions of the students give evidence of the mathematical connection ‘the slope is found with the formula \( \frac{y_2 - y_1}{x_2 - x_1} \)’ and it is strengthened with the arguments given in the interviews. This task was solved correctly by 20 students; they proposed a formula to solve the task and then substituted the coordinates of the given points A and B in this formula. The most common formula is \( m = \frac{y_2 - y_1}{x_2 - x_1} \), followed by \( \frac{y_2 - y_1}{x_2 - x_1} \) and \( P = \frac{y_2 - y_1}{x_2 - x_1} \). For example, Eliott manifests the primary use of the formula to find the slope in a written way (Figure 3) and in the interview (see the corresponding excerpt).

**Interviewer:** How did you solve task 1, the one focused on the slope?

**Eliott:** I know the formula of the slope, and the coordinates of the points are given. This is my data [pointing to the coordinates of the points] and the formula for the slope equals \( y \) two minus \( y \) one over \( x \) two minus \( x \) one, solving this mechanically, we only substitute and get the result.

**Interviewer:** What is the meaning of this result?

**Eliott:** I did not get the idea of the units of the slope, metres or something, I believe I should have used units of measure, but I did not because I can only get the direct result.

The rest of the students also gave similar arguments in the sense that they only used the formula to calculate the slope without an interpretation. For example, they usually say: ‘we have the formula of the slope and two points, so I only do the calculations and get the slope’
Table 2. Procedural type mathematical connections used in the tasks.

<table>
<thead>
<tr>
<th>Task</th>
<th>Formula</th>
<th>Procedures</th>
<th>Partial</th>
<th>Total</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Task 1 Slope</td>
<td>( m = \frac{y_2 - y_1}{x_2 - x_1} )</td>
<td>( m = \frac{9 - 1}{3 - 1} = \frac{8}{2} = 4 )</td>
<td>11</td>
<td>20</td>
<td>60.6%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( y_2 - y_1 ) ( x_2 - x_1 )</td>
<td>7</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( P = \frac{y_2 - y_1}{x_2 - x_1} )</td>
<td>( P = \frac{9 - 1}{3 - 1} = \frac{8}{2} = 4 )</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>Task 2 Velocity</td>
<td>( v = \frac{s_2 - s_1}{t_2 - t_1} )</td>
<td>( v = \frac{15 - 7}{3 - 1} = \frac{8}{2} = 4 )</td>
<td>3</td>
<td>4</td>
<td>12.1%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( v = \frac{h_f - h_i}{t_f - t_i} )</td>
<td>( [8(3) - (3)^2] - [8(1) - (1)^2] )</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( = \frac{15 - 7}{3 - 1} ) ( = \frac{8}{2} = 4 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Task 3 Speed</td>
<td>( r = \frac{y_2 - y_1}{x_2 - x_1} )</td>
<td>( r = \frac{1980 - 1970}{25 - 20} = \frac{15}{5} = 3 )</td>
<td>4</td>
<td>11</td>
<td>33.3%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 1980 - 1970 ) ( 25 - 20 )</td>
<td>10</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R = \frac{P_f - P_i}{t_f - t_i} )</td>
<td>( R = \frac{45 - 30}{1980 - 1970} = \frac{15}{10} = 1.5 )</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( 45 - 30 ) ( 1980 - 1970 )</td>
<td>15</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( R = \frac{N P}{T} )</td>
<td>( 45 - 30 ) ( 1980 - 1970 ) ( 25 - 20 )</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Task 4 Acceleration</td>
<td>( a = \frac{v_2 - v_1}{t_2 - t_1} )</td>
<td>( a = \frac{25 - 15}{3 - 1} = \frac{5}{2} = 2.5 )</td>
<td>5</td>
<td>13</td>
<td>39.4%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( a = \frac{\nu^2 - \nu_1}{2} )</td>
<td>( 25 - 15 ) ( 2 )</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>( a = \frac{\nu}{T} )</td>
<td>( 5 ) ( 2 )</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3. Procedural type mathematical connection to find the slope with the formula \( m = \frac{y_2 - y_1}{x_2 - x_1} \).

or 'I only substitute the values to get the result because I know the formula of the slope’. The justifications of some students that did not solve all tasks to verify their need to associate a formula with the concept of slope, velocity, acceleration and speed. For example, they
say ‘I cannot find it because I don’t remember the formula’, ‘I don’t know if my formula is correct’ or ‘I don’t know the formula to find it’. The procedures and arguments for the rest of the tasks are similar. As shown in Table 2, the students write incorrect formulas to find the acceleration and some of them did not even write a formula in task 4; however, they remember the algorithm, that is to say, they work with the quotient of the differences without considering the meaning of the first formula.

### 4.3. Mathematical connections of the common features type

This category includes all mathematical connections associated with the identification, including the relationship with attributes, properties or qualities shared by the concepts of slope, velocity, speed and acceleration according to the students.

#### 4.3.1. The formula for speed is similar to the formula for slope

Eight students solved task 3 correctly, four of them made the mathematical connection between the common features type between the formulas of slope and speed. These connections are identified in both their written productions and their justifications in the interviews. They write the same formula for the slope and the speed (see Table 2) using some kind of analogy between the slope and the population speed of growth: the $y$ axis of the graph is related with the number of inhabitants and the variable $x$ with time; one of the students ‘mixes’ the variables in the formula for slope to find a formula for speed $R = (y_2 - y_1)/(t_f - t_i)$.

The arguments expressed are ‘the formula for slope helped me in finding the population speed of growth’, ‘I used the formula for slope to find the formula for speed’.

#### 4.3.2. All formulas require the division of the differences

The arguments of the students about the similitudes of the tasks focused on the resemblance between the formulas and their procedures. Five of the 33 students made this mathematical connection (Oscar, Erika, Selene, Eliott and Saraí); some prominent arguments are ‘there is always initial and final data’, ‘two points are needed to solve them’, ‘the variable of time and another variable are always showing up’, referring to the functional situations, ‘substractions are always needed’, ‘they are all quotients’, ‘most of the problems included a substraction between two numbers divided by another substraction between two numbers’. For example, the following arguments appeared during the interview with Erika.

**Interviewer:** What similitudes did you find during the tasks?

**Erika:** I can see the formulas are similar, and the way we used them, because we operate with them first by substituting data and then a substraction and finally we divide by another substraction.

On the other hand, Oscar, another student of this group, made similar arguments:

**Oscar:** There is a close relationship between them, because they are all concerned with the differences of the final coordinates minus the initial coordinates; it is all about interpreting the tasks using the values given in the statement or in the graph to do the procedures and get the result.
The mathematical connections made by the students are clear in these arguments. They made the mathematical connections attending the common features between the formulas and, above all, between the procedures to manipulate the formulas through calculations like quotients and subtraction.

4.3.3. You can find the velocity, acceleration and speed from the formula of the slope

This mathematical connection was identified in the answers to the question about the similitudes and differences in the formulas. Only two students made this mathematical connection. Their written productions lead them to a correct answer in the four tasks; they attributed their success to the similitude of the formulas used (Table 3). Both of them used a mathematical model translated in a formula with a similar structure: a quotient of differences of the form $x_2 - x_1$ or $x_f - x_i$.

Selene said during the interviews, ‘the form $y_2 - y_1$ over $x_2 - x_1$ exists in all the formulas but with different letters and with other names’. She suggested that the formulas had the same structure (quotient of differences) and the particularities depend on the specific variables of each task. On the other hand, Saraí said

‘you can see the same form in all of them (points out the formula for the slope), well you can say that here (referring to the numerator of the formula for the velocity), but it shows us the velocity, we can interchange them, this means that it will always be … literal 2 minus literal 1, for example: $z_2 - z_1$’.

Both students get closer to a connection of the generalization type in this mathematical connection because they expand the formula of the slope to explain the changes of the variables in other formulas, considering that these changes depend on the specific variables of the formulas for velocity, speed and acceleration.

5. Discussion

The categories of the mathematical connections identified in this study are similar to the typology proposed by Businskas [34]. She identified these types: different representations, procedural, common features, inclusion, implication and generalization. We only find coincidences with the first three categories but we also detected evidence of mathematical connections aimed towards generalization.

Our results showed that the most used mathematical connections are of the different representations type, specifically those between the written statement of the task and the algebraic representation through formulas. They were induced by the terms in which the
main request of each task was made (slope, velocity, speed or acceleration). The mathematical connections between graphic and algebraic representations were less numerous even when graphics were present in all tasks.

In particular, for the case of slope our results are consistent to those of Birgin [11] and Deniz and Kabael [12], who found that 8th Grade students had difficulty in moving between the graphical and algebraic representation forms when attempting to find the slope of line and there existed confusion regarding the connection between these aspects. For instance, it was more common for the students to associate the slope with the algebraic representation of the linear function \((y = mx + b)\) instead of its geometric interpretation [11] or to memorize the geometric representation and use it as an algorithm [12]. In our study, these connections could have been used by students to deduce the formulas or as means to remember them using their graphic meaning, but this was not the case. Students preferred to make mathematical connections between the proposed tasks with the algebraic formulas and ignored the graphs. This confirms the reluctance to visualization noticed by Eisenberg and Dreyfus [48] and the preference for formulas to do algorithms and bring into play the procedural connections. Almost two thirds of the participants used them correctly to find the slope and only a third used them in the functional situations. This was possible because they remembered the correct formula and used it to do the necessary procedures, giving priority to the algorithm as reported by Birgin [11], Deniz and Kabael [12], Nagle and Moore-Russo [49] and Wagener [50]. This may, in part, be due to the instruction received by the students; they are prepared to act in this way. Wagener [50] strengthens this hypothesis by claiming that the instruction of the rate of change relies on algorithmic procedures and the application of formulas. Kriek and Koontse [51] found similar results in a study focused on solving problems in Physics. Memorization was a priority for the students, so they only succeeded when the formulas were remembered.

An indicator of the lack of mathematical connections between the slope and the functional situations, along with the answers in the interviews, is the fact that almost two thirds of the students succeeded in the slope task while only a third obtain the correct answers in the functional situations of the velocity, speed and acceleration. Teuscher and Reys [8] reported that the students consider slope, rate of change and inclination as three different and unrelated concepts. Our study widens this result because our participants did not connect the slope with the functional situations of velocity, speed and acceleration. Planinic et al. [16] also found similar results in a study focused on parallel questions in mathematics and physics; the mathematical context included the estimation and graphical interpretation of the slope and the physical context included velocity and acceleration. The students solved the slope tasks correctly but not the velocity and acceleration ones, they did not even recognize the similarities. They attributed these results to their weak ability to transfer the mathematical knowledge to physics; it is, therefore, reasonable to believe that this also happens with our participants.

Almost a fifth of the students were able to make mathematical connections of the common features type when they were asked about the formula and the procedures used; they realized that they always do differences and quotients of differences. Ellis [52] points out that identifying common features is an activity of generalization, but we believe that this specific group of students does not achieve to extend their reasoning beyond this identification of common features. We only found an approach to the connection of generalization
in two of the three students that solved the four tasks correctly. They argued that the same formula for the slope is used, but with the variables changed. This indicates that they are in an abductive phase in the sense of Rivera [53] because they give a hypothesis to explain a given pattern (the formula for the slope) based on the available examples, but they did not achieve the inductive phase of extending the pattern that underlies in the general formula of the rate of change.

The interpretation of the results found by the students in the tasks has only a weak presence. As found by Kriek and Koonse [51] and Lingefjärd and Farahani [54], our participants were able to do the calculus and get the results; however, they did not understand the meaning of those results. For example, almost two thirds of the participants obtain the slope; however, most of them were not able to give any interpretation. Only three students gave an interpretation as ‘rise over run’; this makes a difference from the studies of Stump [55], Deniz and Kabael [56] and Walter and Gerson [57] that claims that this is one of the most used conceptions of slope in teachers and students. We also noticed a trend not to consider the units of measure in the final three tasks, even in the former three students. This result is consistent with the results reported by Herbert and Pierce [23] and Weber and Dorko [58]; they claim that the students do the calculations to find the rate of change with little approach to the interpretation of the meanings of the results in a given context or without considering the units of measure.

6. Conclusions

Our results showed that pre-university students did not identify the relationship between the concepts of rate of change and ratio with the slope of the straight line even when the mathematical curricula considers it beginning in 10th grade [2]. Our findings showed the predominance of the procedural knowledge and little conceptual understanding in the sense of Hiebert and Lefevre [59]. Most of the students did not give interpretations of their results; only 5 of the 33 students made mathematical connections of the common features type, and only 2 of the 33 students made this type of connections with tendencies to generalization. These results may come from the fact that students, teachers and even texts pay more attention to the procedures to find the slope than to the development of the conceptual notions of slope as suggested by Lingefjärd and Farahani [60].

Another possible cause is the attribution of the teachers’ understanding of the concept of rate of change. In this sense, Byerley and Thompson [61] reported that the meanings of slope and rate of change given by the teachers are not enough or work incorrectly and this does not contribute to the emergence of mathematical connections. Coe [62] showed some results that support this hypothesis because he found that some teachers had few connections between the meanings attributed to the slope and the meanings attributed to the constant rate of change, the average rate of change, instantaneous rate of change, slope, and proportionality. In this sense, Diaz [63] revealed no understanding of the rate of change in Mexican elementary school teachers, and not significant in junior high school.

The procedural and conceptual knowledge play an important role in the mathematical connections, they are both positive correlated [64]. This correlation can be developed through instruction and improving the mathematical comprehension. There are causal and bidirectional links among them: improving the procedural knowledge can lead to a better conceptual knowledge and vice versa [65,66]. In this sense, Cho and Nagle [13] attribute
the difficulties in the procedural knowledge of the slope to the lack of its conceptual comprehension; this result strengthens the bi-directionality of both types of knowledge.

Our results strongly suggest the importance of the mathematical connections to improve the understanding of the rate of change. This leads us to different possible future investigations. On the one hand, the identification of the mathematical connections that pre-university mathematics teachers are able to achieve and their relationship to the ones promoted during their courses. On the other hand, the influence of the instruction of the rate of change in a wider context full of mathematical connections with the real world in the improvement of the mathematical understanding.

**Note**

1. Junior High School in Mexico corresponds with the 7th, 8th and 9th grade, pre-university level includes the 10th, 11th and 12th grade, this level is equivalent to Senior High School of the USA.

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**References**


Appendix

Tasks to solve

Dear student: The following tasks aim to explore the mathematical knowledge you developed during pre-university school. Feel free to solve these tasks but write all the procedures you use. Your answers will be treated confidentially. We appreciate your participation.

Task 1. What is the slope of the straight line shown in graphic 1?

Task 2. Graphic 2 represents the position of an object that is moving according to the formula \( s(t) = 8t - t \) (s represents the height in metres and t represents the time in seconds), what is the velocity of the object between the first and the second seconds?
Table A1. Continued.

**Task 3.** Graphic 3 shows the growth of urban and rural population since 1910. What is the speed of growth of the urban population between 1970 and 1980? What is the speed of growth of the urban population growth in the same period?

**Graphic 3.** Population growth in Mexico from 1910 to 2010

**Task 4.** Graphic 4 shows the velocity (in m/s) of a bicycle rider moving in a straight line. What is his acceleration between the first and third seconds?

**Graphic 4.** Representing the velocity of the cyclist