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# The hyperbolicity constant of infinite circulant graphs

DOI 10.1515/math-2017-0061

Received June 21, 2016; accepted March 30, 2017.

**Abstract:** If  $X$  is a geodesic metric space and  $x_1, x_2, x_3 \in X$ , a *geodesic triangle*  $T = \{x_1, x_2, x_3\}$  is the union of the three geodesics  $[x_1x_2]$ ,  $[x_2x_3]$  and  $[x_3x_1]$  in  $X$ . The space  $X$  is  $\delta$ -*hyperbolic* (in the Gromov sense) if any side of  $T$  is contained in a  $\delta$ -neighborhood of the union of the two other sides, for every geodesic triangle  $T$  in  $X$ . Deciding whether or not a graph is hyperbolic is usually very difficult; therefore, it is interesting to find classes of graphs which are hyperbolic. A graph is circulant if it has a cyclic group of automorphisms that includes an automorphism taking any vertex to any other vertex. In this paper we prove that infinite circulant graphs and their complements are hyperbolic. Furthermore, we obtain several sharp inequalities for the hyperbolicity constant of a large class of infinite circulant graphs and the precise value of the hyperbolicity constant of many circulant graphs. Besides, we give sharp bounds for the hyperbolicity constant of the complement of every infinite circulant graph.

**Keywords:** Circulant graph, Gromov hyperbolicity, Geodesics, Infinite graphs

**MSC:** 05C75, 05C12, 05A20

## 1 Introduction

Hyperbolic spaces play an important role in geometric group theory and in the geometry of negatively curved spaces (see [1–3]). The concept of Gromov hyperbolicity grasps the essence of negatively curved spaces like the classical hyperbolic space, simply connected Riemannian manifolds of negative sectional curvature bounded away from 0, and of discrete spaces like trees and the Cayley graphs of many finitely generated groups. It is remarkable that a simple concept leads to such a rich general theory (see [1–3]).

The first works on Gromov hyperbolic spaces deal with finitely generated groups (see [3]). Initially, Gromov spaces were applied to the study of automatic groups in the science of computation (see, e.g., [4]); indeed, hyperbolic groups are strongly geodesically automatic, *i.e.*, there is an automatic structure on the group [5].

The concept of hyperbolicity appears also in discrete mathematics, algorithms and networking. For example, it has been shown empirically in [6] that the internet topology embeds with better accuracy into a hyperbolic space than into a Euclidean space of comparable dimension (formal proofs that the distortion is related to the hyperbolicity can be found in [7]); furthermore, it is evidenced that many real networks are hyperbolic (see, e.g., [8–12]). A few algorithmic problems in hyperbolic spaces and hyperbolic graphs have been considered in recent papers (see [13–16]). Another important application of these spaces is the study of the spread of viruses through the internet (see [17, 18]). Furthermore, hyperbolic spaces are useful in secure transmission of information on the network (see [17–19]); also to traffic flow and effective resistance of networks [20–22].

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In [23] it was proved the equivalence of the hyperbolicity of many negatively curved surfaces and the hyperbolicity of a graph related to it; hence, it is useful to know hyperbolicity criteria for graphs from a geometrical viewpoint. In recent years, the study of mathematical properties of Gromov hyperbolic spaces has become a topic of increasing interest in graph theory and its applications (see, e.g., [11, 17–19, 23–40] and the references therein).

If  $(X, d)$  is a metric space and  $\gamma : [a, b] \rightarrow X$  is a continuous function, we define the *length* of  $\gamma$  as

$$L(\gamma) := \sup \left\{ \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \dots < t_n = b \right\}.$$

We say that a curve  $\gamma : [a, b] \rightarrow X$  is a *geodesic* if we have  $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$  for every  $s, t \in [a, b]$ , where  $L$  and  $d$  denote length and distance, respectively, and  $\gamma|_{[t,s]}$  is the restriction of the curve  $\gamma$  to the interval  $[t, s]$  (then  $\gamma$  is equipped with an arc-length parametrization). The metric space  $X$  is said *geodesic* if for every couple of points in  $X$  there exists a geodesic joining them; we denote by  $[x, y]$  any geodesic joining  $x$  and  $y$ ; this notation is ambiguous, since in general we do not have uniqueness of geodesics, but it is very convenient. Consequently, any geodesic metric space is connected. If the metric space  $X$  is a graph, then the edge joining the vertices  $u$  and  $v$  will be denoted by  $[u, v]$ .

Along the paper we just consider graphs with every edge of length 1. In order to consider a graph  $G$  as a geodesic metric space, identify (by an isometry) any edge  $[u, v] \in E(G)$  with the interval  $[0, 1]$  in the real line; then the edge  $[u, v]$  (considered as a graph with just one edge) is isometric to the interval  $[0, 1]$ . Thus, the points in  $G$  are the vertices and, also, the points in the interior of any edge of  $G$ . In this way, any connected graph  $G$  has a natural distance defined on its points, induced by taking shortest paths in  $G$ , and we can see  $G$  as a metric graph. If  $x, y$  are in different connected components of  $G$ , we define  $d_G(x, y) = \infty$ . Throughout this paper,  $G = (V, E) = (V(G), E(G))$  denotes a simple (without loops and multiple edges) graph (not necessarily connected) such that every edge has length 1 and  $V \neq \emptyset$ . These properties guarantee that any connected component of any graph is a geodesic metric space. Note that to exclude multiple edges and loops is not an important loss of generality, since [27, Theorems 8 and 10] reduce the problem of compute the hyperbolicity constant of graphs with multiple edges and/or loops to the study of simple graphs.

If  $X$  is a geodesic metric space and  $x_1, x_2, x_3 \in X$ , the union of three geodesics  $[x_1, x_2]$ ,  $[x_2, x_3]$  and  $[x_3, x_1]$  is a *geodesic triangle* that will be denoted by  $T = \{x_1, x_2, x_3\}$  and we will say that  $x_1, x_2$  and  $x_3$  are the vertices of  $T$ ; it is usual to write also  $T = \{[x_1, x_2], [x_2, x_3], [x_3, x_1]\}$ . We say that  $T$  is  $\delta$ -thin if any side of  $T$  is contained in the  $\delta$ -neighborhood of the union of the two other sides. We denote by  $\delta(T)$  the sharp thin constant of  $T$ , i.e.  $\delta(T) := \inf\{\delta \geq 0 : T \text{ is } \delta\text{-thin}\}$ . The space  $X$  is  $\delta$ -hyperbolic (or satisfies the *Rips condition* with constant  $\delta$ ) if every geodesic triangle in  $X$  is  $\delta$ -thin. We denote by  $\delta(X)$  the sharp hyperbolicity constant of  $X$ , i.e.  $\delta(X) := \sup\{\delta(T) : T \text{ is a geodesic triangle in } X\}$ . We say that  $X$  is *hyperbolic* if  $X$  is  $\delta$ -hyperbolic for some  $\delta \geq 0$ ; then  $X$  is hyperbolic if and only if  $\delta(X) < \infty$ . If  $X$  has connected components  $\{X_i\}_{i \in I}$ , then we define  $\delta(X) := \sup_{i \in I} \delta(X_i)$ , and we say that  $X$  is hyperbolic if  $\delta(X) < \infty$ .

If we have a triangle with two identical vertices, we call it a *bigon*; note that since this is a special case of the definition, every geodesic bigon in a  $\delta$ -hyperbolic space is  $\delta$ -thin.

In the classical references on this subject (see, e.g., [1, 2, 41]) appear several different definitions of Gromov hyperbolicity, which are equivalent in the sense that if  $X$  is  $\delta$ -hyperbolic with respect to one definition, then it is  $\delta'$ -hyperbolic with respect to another definition (for some  $\delta'$  related to  $\delta$ ). The definition that we have chosen has a deep geometric meaning (see, e.g., [2]).

Trivially, any bounded metric space  $X$  is  $((\text{diam } X)/2)$ -hyperbolic. A normed linear space is hyperbolic if and only if it has dimension one. A geodesic space is 0-hyperbolic if and only if it is a metric tree. If a complete Riemannian manifold is simply connected and its sectional curvatures satisfy  $K \leq c$  for some negative constant  $c$ , then it is hyperbolic. See the classical references [1, 2, 41] in order to find further results.

We want to remark that the main examples of hyperbolic graphs are the trees. In fact, the hyperbolicity constant of a geodesic metric space can be viewed as a measure of how “tree-like” the space is, since those spaces  $X$  with  $\delta(X) = 0$  are precisely the metric trees. This is an interesting subject since, in many applications, one finds that the borderline between tractable and intractable cases may be the tree-like degree of the structure to be dealt with (see, e.g., [42]).

A graph is *circulant* if it has a cyclic group of automorphisms that includes an automorphism taking any vertex to any other vertex. There are large classes of circulant graphs. For instance, every cycle graph, complete graph, crown graph and Möbius ladder is a circulant graph. A complete bipartite graph is a circulant graph if and only if it has the same number of vertices on both sides of its bipartition. A connected finite graph is circulant if and only if it is the Cayley graph of a cyclic group, see [43]. Every circulant graph is a vertex transitive graph and a Cayley graph [44]. It should be noted that Paley graphs is an important class of circulant graph, which is attracting great interest in recent years (see, e.g., [45]).

The circulant is a natural generalization of the double loop network and was first considered by Wong and Coppersmith [46]. Circulant graphs are interesting by the role they play in the design of networks. In the area of computer networks, the standard topology is that of a ring network; that is, a cycle in graph theoretic terms. However, cycles have relatively large diameter, and in an attempt to reduce the diameter by adding edges, we wish to retain certain properties. In particular, we would like to retain maximum connectivity and vertex-transitivity. Hence, most of the earlier research concentrated on using the circulant graphs to build interconnection networks for distributed and parallel systems [47], [48]. The term circulant comes from the nature of its adjacency matrix. A matrix is circulant if all its rows are periodic rotations of the first one. Circulant matrices have been employed for designing binary codes [49]. Theoretical properties of circulant graphs have been studied extensively and surveyed [47].

For a finite graph with  $n$  vertices it is possible to compute  $\delta(G)$  in time  $O(n^{3.69})$  [50] (this is improved in [10, 51]). Given a Cayley graph (of a presentation with solvable word problem) there is an algorithm which allows to decide if it is hyperbolic [52]. However, deciding whether or not a general infinite graph is hyperbolic is usually very difficult. Therefore, it is interesting to relate hyperbolicity with other properties of graphs. The papers [24, 29, 40] prove, respectively, that chordal,  $k$ -chordal and edge-chordal are hyperbolic. Moreover, in [24] it is shown that hyperbolic graphs are path-chordal graphs. These results relating chordality and hyperbolicity are improved in [33]. In the same line, many researches have studied the hyperbolicity of other classes of graphs: complement of graphs [53], vertex-symmetric graphs [54], line graphs [55], bipartite and intersection graphs [56], bridged graphs [32], expanders [22], Cartesian product graphs [57], cubic graphs [58], and random graphs [37–39].

In this paper we prove that infinite circulant graphs and their complements are hyperbolic (see Theorems 2.3, 2.4 and 3.15). We obtain in Theorems 3.7 and 3.8 several sharp inequalities for the hyperbolicity constant of a large class of infinite circulant graphs, and the precise value of the hyperbolicity constant of many circulant graphs. Besides, Theorem 3.15 provides sharp bounds for the hyperbolicity constant of the complement of every infinite circulant graph.

## 2 Every circulant graph is hyperbolic

Let  $(X, d_X)$  and  $(Y, d_Y)$  be two metric spaces. A map  $f : X \rightarrow Y$  is said to be an  $(\alpha, \beta)$ -quasi-isometric embedding, with constants  $\alpha \geq 1$ ,  $\beta \geq 0$  if for every  $x, y \in X$ :

$$\alpha^{-1}d_X(x, y) - \beta \leq d_Y(f(x), f(y)) \leq \alpha d_X(x, y) + \beta.$$

The function  $f$  is  $\varepsilon$ -full if for each  $y \in Y$  there exists  $x \in X$  with  $d_Y(f(x), y) \leq \varepsilon$ .

A map  $f : X \rightarrow Y$  is said to be a quasi-isometry, if there exist constants  $\alpha \geq 1$ ,  $\beta, \varepsilon \geq 0$  such that  $f$  is an  $\varepsilon$ -full  $(\alpha, \beta)$ -quasi-isometric embedding.

A fundamental property of hyperbolic spaces is the following:

**Theorem 2.1** (Invariance of hyperbolicity). *Let  $f : X \rightarrow Y$  be an  $(\alpha, \beta)$ -quasi-isometric embedding between the geodesic metric spaces  $X$  and  $Y$ . If  $Y$  is  $\delta$ -hyperbolic, then  $X$  is  $\delta'$ -hyperbolic, where  $\delta'$  is a constant which just depends on  $\alpha, \beta, \delta$ .*

*Besides, if  $f$  is  $\varepsilon$ -full for some  $\varepsilon \geq 0$  (a quasi-isometry) and  $X$  is  $\delta$ -hyperbolic, then  $Y$  is  $\delta'$ -hyperbolic, where  $\delta'$  is a constant which just depends on  $\alpha, \beta, \delta, \varepsilon$ .*

We denote by  $\text{Aut}(G)$  the set of automorphisms of the graph  $G$  (isomorphisms of  $G$  onto itself). If  $g \in \text{Aut}(G)$  we will denote by  $\langle g \rangle$  the cyclic subgroup of  $\text{Aut}(G)$  generated by  $g$ .

**Definition 2.2.** Let  $G$  be any circulant connected infinite graph and  $g \in \text{Aut}(G)$  with  $\langle g \rangle = \text{Aut}(G)$ . Consider the graph  $G^*$  with  $V(G^*) := V(G)$  and  $E(G^*) := \{[g^n(v^0), g^{n+1}(v^0)]\}_{n \in \mathbb{Z}}$  for some fixed  $v^0 \in V(G)$ . We define

$$n(G, g) := d_G(v^0, g(v^0)),$$

$$N(G, g) := \max \{d_{G^*}(v^0, w) \mid [v^0, w] \in E(G)\}.$$

Note that the definition of  $n(G, g)$  and  $N(G, g)$  do not depend on the choice of  $v^0$ , since  $G$  is a circulant graph. Hence, for every  $v \in V(G)$ ,

$$n(G, g) = d_G(v, g(v)) = d_G(v, g^{-1}(v)),$$

$$N(G, g) = \max \{d_{G^*}(v, w) \mid [v, w] \in E(G)\}.$$

**Theorem 2.3.** Any circulant connected infinite graph  $G$  satisfies the inequality  $\delta(G) \leq c$ , where  $c$  is a constant which just depends on  $n(G, g)$  and  $N(G, g)$ .

*Proof.* For each  $n \in \mathbb{Z}$ , let  $w_n$  be the midpoint of the edge  $[g^n(v^0), g^{n+1}(v^0)] \in E(G^*)$ . Let us define a map  $i : G^* \rightarrow G$  as follows: for each  $n \in \mathbb{Z}$ , let  $i(u) := g^n(v^0)$  for every  $u \in [w_{n-1}w_n] \setminus \{w_{n-1}\}$ . Note that  $i$  is the inclusion map on  $V(G^*) = V(G)$ ; hence,  $i$  is  $(1/2)$ -full.

Fix  $u, v \in V(G) = V(G^*)$ . Let  $u_0 = u, u_1, \dots, u_{k-1}, u_k = v \in V(G)$  with  $d_G(u, v) = \sum_{j=1}^k d_G(u_{j-1}, u_j)$  and  $[u_{j-1}, u_j] \in E(G)$  for every  $1 \leq j \leq k$ ; then we have

$$d_{G^*}(u, v) \leq \sum_{j=1}^k d_{G^*}(u_{j-1}, u_j) \leq \sum_{j=1}^k N(G, g) d_G(u_{j-1}, u_j) = N(G, g) d_G(u, v).$$

Fix  $u, v \in G^*$  and a geodesic  $[uv]$  in  $G^*$ . Recall that  $i(u), i(v) \in V(G) = V(G^*)$ . We have

$$\begin{aligned} d_{G^*}(u, v) &\leq d_{G^*}(u, i(u)) + d_{G^*}(i(u), i(v)) + d_{G^*}(i(v), v) \\ &\leq 1/2 + N(G, g) d_G(i(u), i(v)) + 1/2 \\ &= N(G, g) d_G(i(u), i(v)) + 1, \\ \frac{1}{N(G, g)} d_{G^*}(u, v) &\leq d_G(i(u), i(v)) + \frac{1}{N(G, g)} \leq d_G(i(u), i(v)) + 1. \end{aligned}$$

Fix  $u, v \in V(G) = V(G^*)$ . Let  $v_0 = u, v_1, \dots, v_{r-1}, v_r = v \in V(G)$  with  $d_{G^*}(u, v) = \sum_{j=1}^r d_{G^*}(v_{j-1}, v_j)$  and  $[v_{j-1}, v_j] \in E(G^*)$ ; then  $v_j = g^i(v_{j-1})$  for some  $i \in \{1, -1\}$  and we have

$$d_G(u, v) \leq \sum_{j=1}^r d_G(v_{j-1}, v_j) = \sum_{j=1}^r n(G, g) d_{G^*}(v_{j-1}, v_j) = n(G, g) d_{G^*}(u, v).$$

Fix  $u, v \in G^*$ . We have

$$\begin{aligned} d_G(i(u), i(v)) &\leq n(G, g) d_{G^*}(i(u), i(v)) \\ &\leq n(G, g) (d_{G^*}(i(u), u) + d_{G^*}(u, v) + d_{G^*}(v, i(v))) \\ &\leq n(G, g) (1/2 + d_{G^*}(u, v) + 1/2) \\ &= n(G, g) d_{G^*}(u, v) + n(G, g). \end{aligned}$$

We conclude that  $i$  is a  $(1/2)$ -full  $(\max\{N(G, g), n(G, g)\}, n(G, g))$ -quasi-isometry and Theorem 2.1 gives the result, since  $G^*$  is 0-hyperbolic. □

**Theorem 2.4.** Every circulant graph is hyperbolic.

*Proof.* Let us consider any fixed circulant graph  $G$ . If  $G$  is a finite graph, then it is  $(\text{diam } G)$ -hyperbolic. Assume now that  $G$  is an infinite graph.

If  $G$  is connected, then Theorem 2.3 gives that it is hyperbolic.

If  $G$  is not connected, then it has just a finite number of isomorphic connected components  $G_1, \dots, G_r$ , since  $G$  is a circulant graph; therefore,  $\delta(G) = \max \{\delta(G_1), \dots, \delta(G_r)\} = \delta(G_1)$ . Since  $G_1$  is connected and circulant, Theorem 2.3 gives the result. □

### 3 Bounds for the hyperbolicity constant of infinite circulant graphs

Let  $\{a_1, a_2, \dots, a_k\}$  be a set of integers such that  $0 < a_1 < \dots < a_k$ . We define the circulant graph  $C_\infty(a_1, \dots, a_k)$  as the infinite graph with vertices  $\mathbb{Z}$  and such that  $N(j) = \{j \pm a_i\}_{i=1}^k$  is the set of neighbors of each vertex  $j \in \mathbb{Z}$ . Then  $C_\infty(a_1, \dots, a_k)$  is a regular graph of degree  $2k$ . If  $k = 1$ , then  $C_\infty(1)$  is isometric to the Cayley graph of  $\mathbb{Z}$ , which is 0-hyperbolic. Hence, in what follows we just consider circulant graphs with  $k > 1$ .

Denote by  $J(G)$  the set of vertices and midpoints of edges in  $G$ , and by  $\lfloor t \rfloor$  the lower integer part of  $t$ .

The following results in [25] will be useful.

**Theorem 3.1** ([25, Theorem 2.6]). *For every hyperbolic graph  $G$ ,  $\delta(G)$  is a multiple of  $1/4$ .*

As usual, by *cycle* we mean a simple closed curve, i.e., a path with different vertices, unless the last one, which is equal to the first vertex.

**Theorem 3.2** ([25, Theorem 2.7]). *For any hyperbolic graph  $G$ , there exists a geodesic triangle  $T = \{x, y, z\}$  that is a cycle with  $x, y, z \in J(G)$  and  $\delta(T) = \delta(G)$ .*

We also need the following technical lemmas.

**Lemma 3.3.** *For any integers  $k > 1$  and  $1 < a_2 < \dots < a_k$ , consider  $G = C_\infty(1, a_2, \dots, a_k)$ . Then the following statements are equivalent:*

- (1)  $d_G(0, \lfloor a_k/2 \rfloor) = \lfloor a_k/2 \rfloor$  and, if  $a_k$  is odd, then  $d_G(0, \lfloor a_k/2 \rfloor + 1) \geq \lfloor a_k/2 \rfloor$ .
- (2)  $a_2 \geq a_k - 1$ .
- (3) We have either  $k = 2$  or  $k = 3$  and  $a_2 = a_3 - 1$ .

*Proof.* Assume that (1) holds. If  $k = 2$ , then (2) holds; hence, we can assume  $k \geq 3$ . Define  $r := \lfloor a_k/2 \rfloor$ . If  $a_2 \leq r$ , then  $r = d_G(0, r)$  by hypothesis and  $r = d_G(0, r) \leq d_G(0, a_2) + d_G(a_2, r) \leq 1 + r - a_2 < r$ , which is a contradiction. Thus we conclude  $a_2 > r$ . Therefore,

$$r = d_G(0, r) \leq d_G(0, a_2) + d_G(a_2, r) \leq 1 + a_2 - r,$$

and  $a_2 \geq 2r - 1$ . Hence,  $a_2 \geq a_k - 1$  if  $a_k$  is even. Since  $a_2 > r$ , if  $a_k$  is odd, then

$$r \leq d_G(0, r + 1) \leq d_G(0, a_2) + d_G(a_2, r + 1) \leq 1 + a_2 - (r + 1),$$

and  $a_2 \geq 2r = a_k - 1$ . Then (2) holds.

A simple computation provides the converse implication.

The equivalence of (2) and (3) is elementary. □

Let us define the subset  $\mathcal{E}$  of infinite circulant graphs as  $\mathcal{E} := \{C_\infty(1, 2m + 1, 2m + 2, 2m + 3)\}_{m \geq 1}$ .

**Lemma 3.4.** *Consider any integers  $k > 1$  and  $1 < a_2 < \dots < a_k$  with  $a_2 < a_k - 1$ , and  $G = C_\infty(1, a_2, \dots, a_k) \notin \mathcal{E}$ . Then*

$$\min \{d_G(0, u), d_G(a_j, u)\} \leq \left\lfloor \frac{a_k}{2} \right\rfloor - 1, \quad (1)$$

for every  $u \in \mathbb{Z}$  with  $0 \leq u \leq a_j$  and  $1 \leq j \leq k$ .

*Proof.* Since  $a_2 < a_k - 1$ , Lemma 3.3 gives:

- (i) if  $a_k$  is even, then  $d_G(0, \lfloor a_k/2 \rfloor) < \lfloor a_k/2 \rfloor$ ,
- (ii) if  $a_k$  is odd, then  $d_G(0, \lfloor a_k/2 \rfloor) < \lfloor a_k/2 \rfloor$  or  $d_G(0, \lfloor a_k/2 \rfloor + 1) < \lfloor a_k/2 \rfloor$ .

If  $d_G(0, \lfloor a_k/2 \rfloor) < \lfloor a_k/2 \rfloor$ , then inequality (1) trivially holds. Hence, we can assume that  $a_k$  is odd,  $d_G(0, \lfloor a_k/2 \rfloor) = \lfloor a_k/2 \rfloor$  and  $d_G(0, \lfloor a_k/2 \rfloor + 1) < \lfloor a_k/2 \rfloor$ . These facts imply that  $d_G(0, \lfloor a_k/2 \rfloor + 1) = \lfloor a_k/2 \rfloor - 1$ .

If  $a_2 \leq \lfloor a_k/2 \rfloor$ , then  $d_G(0, \lfloor a_k/2 \rfloor) < \lfloor a_k/2 \rfloor$ , which is a contradiction. Therefore,  $a_2 > \lfloor a_k/2 \rfloor$  and

$$\left\lfloor \frac{a_k}{2} \right\rfloor - 1 = d_G\left(0, \left\lfloor \frac{a_k}{2} \right\rfloor + 1\right) \leq 1 + a_2 - \left\lfloor \frac{a_k}{2} \right\rfloor - 1, \quad a_k - 2 \leq a_2 < a_k - 1.$$

We conclude  $a_2 = a_k - 2$ , and  $k = 3$  or  $k = 4$ . If  $k = 4$ , then  $a_2 = a_4 - 2$ ,  $a_3 = a_4 - 1$  and  $G \in \mathcal{E}$ , since  $a_k$  is odd. This is a contradiction, and we conclude  $k = 3$  and  $a_2 = a_3 - 2$ .

If  $j = 1$ , then  $\min\{d_G(0, u), d_G(1, u)\} = 0$  for every  $0 \leq u \leq 1$ .

If  $j = 2$ , then for every  $0 \leq u \leq a_2$

$$\min\{d_G(0, u), d_G(a_2, u)\} \leq \min\{u, a_2 - u\} \leq \left\lfloor \frac{a_2}{2} \right\rfloor = \left\lfloor \frac{a_3}{2} \right\rfloor - 1.$$

If  $j = 3$ , then  $d_G(\lfloor a_3/2 \rfloor, a_3) = d_G(0, \lfloor a_3/2 \rfloor + 1) = \lfloor a_3/2 \rfloor - 1$ , and for every  $0 \leq u \leq a_3$  with  $u \neq \lfloor a_3/2 \rfloor, \lfloor a_3/2 \rfloor + 1$ ,

$$\min\{d_G(0, u), d_G(a_3, u)\} \leq \min\{u, a_3 - u\} \leq \left\lfloor \frac{a_3}{2} \right\rfloor - 1.$$

This finishes the proof. □

**Proposition 3.5.** Consider any integers  $k > 1$  and  $1 < a_2 < \dots < a_k$ . If we have either  $k = 2$ , or  $k = 3$  and  $a_2 = a_3 - 1$ , or  $C_\infty(1, a_2, \dots, a_k) \in \mathcal{E}$ , then

$$\delta(C_\infty(1, a_2, \dots, a_k)) \geq \frac{1}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor.$$

*Proof.* Define  $r := \lfloor a_k/2 \rfloor$  and  $G := C_\infty(1, a_2, \dots, a_k)$ .

Assume first that we have either  $k = 2$ , or  $k = 3$  and  $a_2 = a_3 - 1$ . Consider the curves  $\gamma_1, \gamma_2$  in  $G$  joining  $x := 0$  and  $y := r + (r + 1)a_k$  given by

$$\begin{aligned} \gamma_1 &:= [0, 1] \cup [1, 2] \cup \dots \cup [r - 1, r] \cup [r, r + a_k] \cup [r + a_k, r + 2a_k] \cup \dots \cup [r + ra_k, r + (r + 1)a_k], \\ \gamma_2 &:= [0, a_k] \cup [a_k, 2a_k] \cup \dots \cup [ra_k, (r + 1)a_k] \cup [(r + 1)a_k, (r + 1)a_k + 1] \cup \dots \\ &\quad \dots \cup [(r + 1)a_k + 1, (r + 1)a_k + 2] \cup [r + (r + 1)a_k - 1, r + (r + 1)a_k]. \end{aligned}$$

Lemma 3.3 gives that  $\gamma_1$  and  $\gamma_2$  are geodesics; then  $d_G(x, y) = L(\gamma_1) = L(\gamma_2) = 2r + 1$ . Let  $T$  be the geodesic bigon  $T = \{\gamma_1, \gamma_2\}$  in  $G$ . If  $p$  is the midpoint of  $[r, r + a_k]$ , then  $d_G(p, x) = d_G(p, y) = r + 1/2$  and Lemma 3.3 gives that  $d_G(p, \gamma_2) = r + 1/2$ . Hence,

$$\delta(C_\infty(1, a_2, \dots, a_k)) \geq d_G(p, \gamma_2) = \frac{1}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor.$$

Assume now that  $C_\infty(1, a_2, \dots, a_k) \in \mathcal{E}$ . Note that  $r + ra_k = (r + 1)a_{k-1}$ , since  $a_k = a_{k-1} + 1$  is odd. Consider the curves  $\gamma_1, \gamma_2, \gamma_3$  in  $G$

$$\begin{aligned} \gamma_1 &:= [-r - ra_k, -r - (r - 1)a_k] \cup \dots \cup [-r - 2a_k, -r - a_k] \cup [-r - a_k, -r] \cup [-r, r] \\ &\quad \cup [r, r + a_k] \cup [r + a_k, r + 2a_k] \cup \dots \cup [r + (r - 1)a_k, r + ra_k], \\ \gamma_2 &:= [-(r + 1)a_{k-1}, -ra_{k-1}] \cup \dots \cup [-2a_{k-1}, -a_{k-1}] \cup [-a_{k-1}, 0], \\ \gamma_3 &:= [0, a_{k-1}] \cup [a_{k-1}, 2a_{k-1}] \cup \dots \cup [ra_{k-1}, (r + 1)a_{k-1}], \end{aligned}$$

joining  $x := -(r + 1)a_{k-1}$ ,  $y := (r + 1)a_{k-1}$  and  $z := 0$ . One can check that  $\gamma_1, \gamma_2$  and  $\gamma_3$  are geodesics in  $G$ . Let  $T$  be the geodesic triangle  $T = \{\gamma_1, \gamma_2, \gamma_3\}$ . Note that  $d_G(0, r) = r$ , since  $G \in \mathcal{E}$ . Therefore, if  $p$  is the midpoint of  $[-r, r]$ , then

$$\delta(G) \geq d_G(p, \gamma_2 \cup \gamma_3) = \frac{1}{2} + d_G(r, \gamma_3) = \frac{1}{2} + d_G(r, \{0, 2r\}) = \frac{1}{2} + r = \frac{1}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor. \quad \square$$

Let  $G = C_\infty(1, a_2, \dots, a_k)$ . If  $x \in V(G)$ , we define  $x_1 := x_2 := x$ ; if  $x \in G \setminus V(G)$ , we define  $x_1$  and  $x_2$  as the endpoints of the edge containing  $x$  with  $x_1 < x_2$ . Therefore,  $1 \leq x_2 - x_1 \leq a_k$  if  $x \in G \setminus V(G)$ . If  $x, y \in J(G)$ , we say that  $xL y$  if  $x = y$  or  $x_2 \leq y_1$ , and  $x$  and  $y$  are *related* (and we write  $xR y$ ) if  $xL y$  or  $yL x$ . Note that  $L$  is an order relation on  $J(G)$ .

**Lemma 3.6.** Consider any integers  $k > 1$  and  $1 < a_2 < \dots < a_k$ . If  $x, y \in J(C_\infty(1, a_2, \dots, a_k))$  and  $x$  and  $y$  are not related, then

$$d_G(x, y) \leq 1 + \left\lfloor \frac{a_k}{2} \right\rfloor.$$

Furthermore, if  $a_2 < a_k - 1$  and  $C_\infty(1, a_2, \dots, a_k) \notin \mathcal{E}$ , then

$$d_G(x, y) \leq \left\lfloor \frac{a_k}{2} \right\rfloor.$$

*Proof.* Let  $G = C_\infty(1, a_2, \dots, a_k)$ . If  $x_1 \leq y_1 \leq y_2 \leq x_2$ , then the cycle

$$\begin{aligned} \sigma := & [x_1, x_1 + 1] \cup [x_1 + 1, x_1 + 2] \cup \dots \cup [y_1 - 1, y_1] \cup [y_1, y_2] \cup [y_2, y_2 + 1] \cup \\ & \dots \cup [x_2 - 2, x_2 - 1] \cup [x_2 - 1, x_2] \cup [x_2, x_1] \end{aligned}$$

has length at most  $1 + a_k$ . Since  $\sigma$  is a cycle containing the points  $x$  and  $y$ , we have

$$d_G(x, y) \leq \frac{1}{2} L(\sigma) \leq \frac{1 + a_k}{2} \leq 1 + \left\lfloor \frac{a_k}{2} \right\rfloor.$$

If  $a_2 < a_k - 1$  and  $G \notin \mathcal{E}$ , then Lemma 3.4 gives  $d_G(x, y_1) \leq 1/2 + \lfloor a_k/2 \rfloor - 1 = \lfloor a_k/2 \rfloor - 1/2$  and  $d_G(x, y) \leq \lfloor a_k/2 \rfloor$ .

If  $y_1 \leq x_1 \leq x_2 \leq y_2$ , then the same argument gives these inequalities.

If  $x_1 \leq y_1 < x_2 \leq y_2$ , then  $x_2 - y_1 + y_1 - x_1 \leq a_k$  and  $\min\{x_2 - y_1, y_1 - x_1\} \leq \lfloor a_k/2 \rfloor$ . If  $x_2 - y_1 \leq \lfloor a_k/2 \rfloor$ , then  $d_G(x, y) \leq d_G(x, x_2) + x_2 - y_1 + d_G(y_1, y) \leq 1 + \lfloor a_k/2 \rfloor$ . If  $y_1 - x_1 \leq \lfloor a_k/2 \rfloor$ , then  $d_G(x, y) \leq d_G(x, x_1) + y_1 - x_1 + d_G(y_1, y) \leq 1 + \lfloor a_k/2 \rfloor$ . If  $a_2 < a_k - 1$  and  $C_\infty(1, a_2, \dots, a_k) \notin \mathcal{E}$ , then Lemma 3.4 also gives  $d_G(x, y_1) \leq \lfloor a_k/2 \rfloor - 1/2$  and  $d_G(x, y) \leq \lfloor a_k/2 \rfloor$ .

If  $y_1 \leq x_1 < y_2 \leq x_2$ , then the same argument gives these inequalities. □

We can state now the main result of this section, which provides a sharp upper bound for the hyperbolicity constant of infinite circulant graphs.

**Theorem 3.7.** For any integers  $k > 1$  and  $1 < a_2 < \dots < a_k$ ,

$$\delta(C_\infty(1, a_2, \dots, a_k)) \leq \frac{1}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor,$$

and the equality is attained if and only if we have either  $k = 2$ , or  $k = 3$  and  $a_2 = a_3 - 1$ , or  $k = 4$ ,  $a_2 = a_4 - 2$ ,  $a_3 = a_4 - 1$  and  $a_4$  is odd.

*Proof.* In order to bound  $\delta(G)$  with  $G = C_\infty(1, a_2, \dots, a_k)$ , let us consider a geodesic triangle  $T = \{x, y, z\}$  in  $G$  and  $p \in [xy]$ ; by Theorem 3.2 we can assume that  $T$  is a cycle with  $x, y, z \in J(G)$ . We consider several cases.

(A) If  $x$  and  $y$  are not related, then Lemma 3.6 gives

$$d_G(p, [xz] \cup [yz]) \leq d_G(p, \{x, y\}) \leq \frac{1}{2} d_G(x, y) \leq \frac{1}{2} + \frac{1}{2} \left\lfloor \frac{a_k}{2} \right\rfloor. \tag{2}$$

(B) Assume that  $xRy$ . Without loss of generality we can assume that  $xLy$ . Denote by  $w_1, \dots, w_m$  the vertices in  $[xy]$  such that  $w_1 \in \{x_1, x_2\}$ ,  $w_m \in \{y_1, y_2\}$  and  $d_G(w_j, w_{j+1}) = 1$  for every  $1 \leq j < m$ ; we define

$$i_0 := \min\{1 \leq i \leq m \mid w_j \geq x_2 \ \forall i \leq j \leq m\}, \quad x_0 := w_{i_0}.$$

(B.1) Assume that either  $x \in V(G)$  or  $x$  is the midpoint of an edge  $[r, r + 1] \in E(G)$  for some  $r \in \mathbb{Z}$ . If  $x_0 = x_2$ , then  $L([xx_0]) \leq 1/2$ . If  $x_0 > x_2$ , then the cycle

$$\sigma := [w_{i_0}, w_{i_0-1}] \cup [w_{i_0-1}, w_{i_0-1} + 1] \cup [w_{i_0-1} + 1, w_{i_0-1} + 2] \cup \dots \cup [w_{i_0} - 2, w_{i_0} - 1] \cup [w_{i_0} - 1, w_{i_0}]$$

has length at most  $1 + a_k$ . Since  $x \in V(G)$  or  $x$  is the midpoint of an edge  $[r, r + 1]$ , thus  $\sigma$  is a cycle containing the points  $x$  and  $x_0$ , and we have

$$L([xx_0]) = d_G(x, x_0) \leq \frac{1}{2} L(\sigma) \leq \frac{1 + a_k}{2} < \frac{3}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor.$$

(B.2) Assume now that  $x$  is the midpoint of an edge  $[r, r + a_j] \in E(G)$  for some  $r \in \mathbb{Z}$  and  $1 < j \leq k$ . If  $x_0 = x_2$ , then  $L([xx_0]) = 1/2$ . If  $w_{i_0-1} = x_1$ , then  $L([xx_0]) = 3/2$ . If  $x_0 > x_2$  and  $w_{i_0-1} \neq x_1$ , then we have either  $w_{i_0-1} < x_1 < x_2 < w_{i_0}$  or  $x_1 < w_{i_0-1} < x_2 < w_{i_0}$ . In the first case the cycle

$$\sigma := [w_{i_0}, w_{i_0-1}] \cup [w_{i_0-1}, w_{i_0-1} + 1] \cup \dots \cup [x_1 - 1, x_1] \cup [x_1, x_2] \cup [x_2, x_2 + 1] \cup \dots \cup [w_{i_0} - 1, w_{i_0}]$$

has length at most  $a_k$ . Since  $\sigma$  is a cycle containing the points  $x$  and  $x_0$ , we have

$$L([xx_0]) = d_G(x, x_0) \leq \frac{1}{2} L(\sigma) \leq \frac{a_k}{2}.$$

Assume that  $x_1 < w_{i_0-1} < x_2 < w_{i_0}$ . Then

$$\begin{aligned} L([xx_0]) = d_G(x, x_0) &\leq \frac{1}{2} + \min\{d_G(x_2, w_{i_0-1}), d_G(w_{i_0-1}, x_1)\} + 1 \\ &\leq \frac{3}{2} + \min\{x_2 - w_{i_0-1}, w_{i_0-1} - x_1\} \leq \frac{3}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor. \end{aligned}$$

Therefore, in Case (B), if  $p \in [xx_0] \setminus B(x_0, 3/4)$ , then

$$d_G(p, [xz] \cup [yz]) \leq d_G(p, x) \leq L([xx_0] \setminus B(x_0, 3/4)) = L([xx_0]) - \frac{3}{4} \leq \frac{3}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor. \tag{3}$$

Define

$$j_0 := \max\{1 \leq i \leq m \mid w_j \leq y_1 \ \forall 1 \leq j \leq i\}, \quad y_0 := w_{j_0}.$$

A similar argument to the previous one shows that if  $p \in [y_0y] \setminus B(y_0, 3/4)$ , then

$$d_G(p, [xz] \cup [yz]) \leq \frac{3}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor. \tag{4}$$

Since  $T$  is a continuous curve, we have

$$([xz] \cup [yz]) \cap \{x_1, x_2\} \neq \emptyset, \quad ([xz] \cup [yz]) \cap \{y_1, y_2\} \neq \emptyset.$$

(a) Assume that  $y_0 \in [xx_0] \setminus \{x_0\}$ .

(a.1) If  $d_G(x_0, y_0) \geq 2$ , then

$$\begin{aligned} L([xy]) = L([xx_0]) + L([y_0y]) - L([y_0x_0]) &\leq \frac{3}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor + \frac{3}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor - 2 = 1 + 2 \left\lfloor \frac{a_k}{2} \right\rfloor, \\ d_G(p, [xz] \cup [yz]) \leq d_G(p, \{x, y\}) &\leq \frac{1}{2} L([xy]) \leq \frac{1}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor. \end{aligned}$$

(a.2) If  $d_G(x_0, y_0) = 1$ , then  $w_{j_0} = y_0 = w_{i_0-1}$ ,  $w_{i_0} = x_0 = w_{j_0+1}$  and the definitions of  $x_0$  and  $y_0$  give  $y_0 < x_2 \leq x_0$  and  $y_0 \leq y_1 < x_0$ . Since  $[x_0, y_0] \in E(G)$ ,

$$\begin{aligned} d_G(x_2, y_1) &\leq \frac{1 + a_k}{2}, \\ L([xy]) = d_G(x, y) &\leq d_G(x, x_2) + d_G(x_2, y_1) + d_G(y_1, y) \leq \frac{1}{2} + \frac{1 + a_k}{2} + \frac{1}{2} = \frac{3 + a_k}{2}, \\ d_G(p, [xz] \cup [yz]) &\leq d_G(p, \{x, y\}) \leq \frac{1}{2} L([xy]) \leq \frac{3 + a_k}{4} \leq \frac{1}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor. \end{aligned}$$

(b) Assume that  $y_0 \notin [xx_0] \setminus \{x_0\}$ . Since  $x_2 \leq x_0$ ,  $y_0 \leq y_1$  and  $[xz] \cup [yz]$  is a continuous curve joining  $x$  and  $y$ , if  $p \in V(G) \cap [x_0y_0] \subset [xy]$ , then there exist  $u, v \in V(G) \cap ([xz] \cup [yz])$  with  $[u, v] \in E(G)$  and  $u \leq p \leq v$ . Since  $T$  is a cycle, we have

$$d_G(p, [xz] \cup [yz]) \leq d_G(p, \{u, v\}) \leq \min\{p - u, v - p\} \leq \left\lfloor \frac{a_k}{2} \right\rfloor.$$

Therefore, if  $p \in [x_0y_0] \cup B(x_0, 3/4) \cup B(y_0, 3/4)$ , then

$$d_G(p, [xz] \cup [yz]) < \frac{3}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor.$$



This inequality, (2), (3) and (4) give

$$\delta(C_\infty(1, a_2, \dots, a_k)) \leq \frac{3}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor. \quad (5)$$

By Theorem 3.1, in order to finish the proof it suffices to show that the equality in (5) is not attained. Seeking for a contradiction, assume that the equality is attained. The proof of (5) gives that we have

$$d_G(p, [xz] \cup [yz]) = \frac{3}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor,$$

where  $p$  is the point in  $[xx_0]$  with  $d_G(p, x_0) = 3/4$  or the point in  $[y_0y]$  with  $d_G(p, y_0) = 3/4$ . By symmetry, without loss of generality we can assume that  $p$  is the point in  $[xx_0]$  with  $d_G(p, x_0) = 3/4$ ; thus  $d_G(p, w_{i_0-1}) = 1/4$ . Therefore, we are in Case (B.2) with  $x_1 < w_{i_0-1} < x_2 < w_{i_0}$  and

$$L([xx_0]) = \frac{3}{2} + \min\{d_G(x_2, w_{i_0-1}), d_G(w_{i_0-1}, x_1)\} = \frac{3}{2} + \min\{x_2 - w_{i_0-1}, w_{i_0-1} - x_1\} = \frac{3}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor.$$

Then

$$\min\{d_G(x_2, w_{i_0-1}), d_G(w_{i_0-1}, x_1)\} = \min\{x_2 - w_{i_0-1}, w_{i_0-1} - x_1\} = \left\lfloor \frac{a_k}{2} \right\rfloor,$$

and we conclude

$$\left\lfloor \frac{a_k}{2} \right\rfloor \leq x_2 - w_{i_0-1}, w_{i_0-1} - x_1 \leq \left\lfloor \frac{a_k}{2} \right\rfloor + 1. \quad (6)$$

(I) Assume that  $x_2 \in [xx_0]$ . Since  $T$  is a cycle, then  $x_1 \in [xz] \cup [yz]$  and  $x_1 < w_{i_0-1}$ . Hence, since  $[xz] \cup [yz]$  is a continuous curve joining  $x$  and  $y$ , and  $d_G(p, w_{i_0-1}) = 1/4$ , we obtain as above

$$\frac{3}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor = d_G(p, [xz] \cup [yz]) \leq d_G(p, w_{i_0-1}) + d_G(w_{i_0-1}, [xz] \cup [yz]) \leq \frac{1}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor,$$

which is a contradiction.

(II) Assume that  $x_1 \in [xx_0]$ .

(II.1) If  $x_2 - x_1 = x_0 - w_{i_0-1} = a_k$ , then  $w_{i_0-1} - x_1 = x_0 - x_2$ ,  $d_G(x_1, w_{i_0-1}) = d_G(x_2, x_0)$  and

$$\begin{aligned} d_G(x, x_0) &= d_G(x, x_1) + d_G(x_1, w_{i_0-1}) + d_G(w_{i_0-1}, x_0) > d_G(x, x_1) + d_G(x_1, w_{i_0-1}) \\ &= \frac{1}{2} + d_G(x_2, x_0) = d_G(x, x_2) + d_G(x_2, x_0) \geq d_G(x, x_0), \end{aligned}$$

which is a contradiction.

(II.2) If  $x_2 - x_1 < a_k$ , then (6) gives

$$x_2 - w_{i_0-1} = w_{i_0-1} - x_1 = \left\lfloor \frac{a_k}{2} \right\rfloor,$$

and

$$\frac{3}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor = d_G(p, [xz] \cup [yz]) \leq d_G(p, x_2) \leq d_G(p, w_{i_0-1}) + d_G(w_{i_0-1}, x_2) \leq \frac{1}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor,$$

which is a contradiction.

(II.3) Assume  $x_2 - x_1 = a_k$  and  $x_0 - w_{i_0-1} < a_k$ . Since  $x_1 \in [xx_0]$ , we have  $w_{i_0-1} - x_1 = \lfloor a_k/2 \rfloor$ .

(II.3.1) If  $x_2 - w_{i_0-1} = \lfloor a_k/2 \rfloor$ , then

$$\frac{3}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor = d_G(p, [xz] \cup [yz]) \leq d_G(p, x_2) \leq d_G(p, w_{i_0-1}) + d_G(w_{i_0-1}, x_2) \leq \frac{1}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor,$$

which is a contradiction.

(II.3.2) If  $x_2 - w_{i_0-1} = \lfloor a_k/2 \rfloor + 1$ , then  $a_k = x_2 - x_1 = 2\lfloor a_k/2 \rfloor + 1$  and

$$\begin{aligned} d_G(x_0, x_2) &\leq x_0 - x_2 = x_0 - w_{i_0-1} - (x_2 - w_{i_0-1}) < a_k - \left\lfloor \frac{a_k}{2} \right\rfloor - 1 = \left\lfloor \frac{a_k}{2} \right\rfloor, \\ \frac{3}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor &= d_G(p, [xz] \cup [yz]) \leq d_G(p, x_2) \leq d_G(p, x_0) + d_G(x_0, x_2) < \frac{3}{4} + \left\lfloor \frac{a_k}{2} \right\rfloor, \end{aligned}$$

which is a contradiction.

This finishes the proof of the inequality.

If we have either  $k = 2$ , or  $k = 3$  and  $a_2 = a_3 - 1$ , or  $k = 4$ ,  $a_2 = a_4 - 2$ ,  $a_3 = a_4 - 1$  and  $a_4$  is odd, then Proposition 3.5 gives

$$\delta(C_\infty(1, a_2, \dots, a_k)) \geq \frac{1}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor.$$

Since we have the converse inequality, we conclude that the equality holds.

Assume now that the equality holds. Denote by  $G$  the circulant graph  $C_\infty(1, a_2, \dots, a_k)$ . By Theorem 3.2 there exist a geodesic triangle  $T = \{x, y, z\}$  in  $G$  that is a cycle with  $x, y, z \in J(G)$ , and  $p \in [xy]$  with  $d_G(p, [xz] \cup [yz]) = 1/2 + \lfloor a_k/2 \rfloor$ . Hence,  $p \in J(G)$ .

Seeking for a contradiction, assume that  $a_2 < a_k - 1$  and that  $G \notin \mathcal{E}$ .

If  $x$  and  $y$  are not related, then Lemma 3.6 gives  $d_G(x, y) \leq \lfloor a_k/2 \rfloor$ . Since  $d_G(x, y) = d_G(x, p) + d_G(p, y) \geq 1 + 2\lfloor a_k/2 \rfloor$ , thus  $x$  and  $y$  are related, and without loss of generality we can assume  $xL y$ . Since  $d_G(p, x), d_G(p, y) \geq 1/2 + \lfloor a_k/2 \rfloor$ , the previous argument gives that  $x$  and  $p$  are related, and  $p$  and  $y$  are related.

(i) Assume first  $xL p$  and  $pL y$ .

Since  $p_1 - x_1 \geq p_1 - x_2 \geq d_G(x_2, p_1) \geq \lfloor a_k/2 \rfloor - 1/2$ , we have  $p_1 - x_1 \geq p_1 - x_2 \geq \lfloor a_k/2 \rfloor$ , and there is some point  $u \in ([xz] \cup [yz]) \cap V(G)$  with  $u \leq p_1$ . A similar argument gives that there is some point  $v \in ([xz] \cup [yz]) \cap V(G)$  with  $p_2 \leq v$ . Denote by  $u$  and  $v$  any couple of vertices satisfying these properties. We are going to prove that  $v - u > a_k$ .

If  $p \in V(G)$ , then

$$\begin{aligned} \min \{p - u, v - p\} &\geq \min \{d_G(u, p), d_G(p, v)\} \geq \frac{1}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor, \\ \min \{p - u, v - p\} &\geq 1 + \left\lfloor \frac{a_k}{2} \right\rfloor, \quad v - u \geq 2 + 2\left\lfloor \frac{a_k}{2} \right\rfloor > a_k. \end{aligned}$$

If  $p$  is the midpoint of some edge  $[m, m + a_j]$  with  $m \in \mathbb{Z}$  and  $2 \leq j \leq k$ , then Lemma 3.4 gives  $d_G(p, w) \leq \lfloor a_k/2 \rfloor - 1/2 < 1/2 + \lfloor a_k/2 \rfloor$  for every  $w \in \mathbb{Z}$  with  $m \leq w \leq m + a_j$ , since  $a_2 < a_k - 1$  and  $G \notin \mathcal{E}$ . Furthermore,

$$\begin{aligned} \min \{m - u, v - m - a_j\} &\geq \min \{d_G(u, m), d_G(m + a_j, v)\} \geq \left\lfloor \frac{a_k}{2} \right\rfloor, \\ v - u &\geq a_j + 2\left\lfloor \frac{a_k}{2} \right\rfloor \geq 2 + 2\left\lfloor \frac{a_k}{2} \right\rfloor > a_k. \end{aligned}$$

Finally, assume that  $p$  is the midpoint of some edge  $[m, m + 1]$  with  $m \in \mathbb{Z}$ . By Lemma 3.3, we have  $d_G(0, \lfloor a_k/2 \rfloor) \leq \lfloor a_k/2 \rfloor - 1$  or  $d_G(0, \lfloor a_k/2 \rfloor + 1) \leq \lfloor a_k/2 \rfloor - 1$ . Therefore,

$$\begin{aligned} d_G\left(p, p \pm \left(\left\lfloor \frac{a_k}{2} \right\rfloor + \frac{1}{2}\right)\right) &\leq \left\lfloor \frac{a_k}{2} \right\rfloor - \frac{1}{2}, \quad \min \{p - u, v - p\} \geq \frac{3}{2} + \left\lfloor \frac{a_k}{2} \right\rfloor, \\ v - u &\geq 3 + 2\left\lfloor \frac{a_k}{2} \right\rfloor \geq 2 + 2\left\lfloor \frac{a_k}{2} \right\rfloor > a_k. \end{aligned}$$

Hence, we have proved  $v - u > a_k$  in every case.

Since if  $p$  is the midpoint of some edge  $[m, m + a_j]$  with  $m \in \mathbb{Z}$  and  $2 \leq j \leq k$  then  $d_G(p, w) < 1/2 + \lfloor a_k/2 \rfloor$  for every  $w \in \mathbb{Z}$  with  $m \leq w \leq m + a_j$ , and  $[xz] \cup [yz]$  is a continuous curve joining  $u$  and  $v$ , there exist  $u_0, v_0 \in ([xz] \cup [yz]) \cap V(G)$  with  $u_0 \leq p_1, p_2 \leq v_0$  and  $[u_0, v_0] \in E(G)$ . Hence,  $a_k \geq v_0 - u_0 > a_k$ , which is a contradiction.

(ii) Assume now that  $pL x$  or  $yL p$ . By symmetry, we can assume that  $pL x$  and thus  $p \in [xx_0]$  and  $d_G(p, x_0) \geq 1$ . Since  $d_G(x, x_0) \leq 3/2 + \lfloor a_k/2 \rfloor$ , we have  $1/2 + \lfloor a_k/2 \rfloor = d_G(p, [xz] \cup [yz]) \leq d_G(x, p) \leq 1/2 + \lfloor a_k/2 \rfloor$ . Therefore,  $d_G(x, p) = 1/2 + \lfloor a_k/2 \rfloor$ ,  $d_G(p, x_0) = 1$ ,  $p \in V(G)$  and  $x$  is the midpoint of  $[x_1, x_2] \in E(G)$ . Thus  $d_G(p, x_1) \geq \lfloor a_k/2 \rfloor$ ,  $d_G(p, x_2) \geq \lfloor a_k/2 \rfloor$  and  $p \leq x_1 \leq x_2 \leq x_0$ . By Lemma 3.3, we have  $d_G(0, \lfloor a_k/2 \rfloor) \leq \lfloor a_k/2 \rfloor - 1$  or  $d_G(0, \lfloor a_k/2 \rfloor + 1) \leq \lfloor a_k/2 \rfloor - 1$ .

If  $d_G(0, \lfloor a_k/2 \rfloor) \leq \lfloor a_k/2 \rfloor - 1$ , then  $x_1 \geq \lfloor a_k/2 \rfloor + 1, x_2 \leq a_k - (\lfloor a_k/2 \rfloor - 1)$  and

$$1 \leq x_2 - x_1 \leq a_k - (\lfloor a_k/2 \rfloor - 1) - \lfloor a_k/2 \rfloor - 1 = a_k - 2\lfloor a_k/2 \rfloor \leq 0,$$

which is a contradiction.

If  $d_G(0, \lfloor a_k/2 \rfloor + 1) \leq \lfloor a_k/2 \rfloor - 1$ , then  $x_2 \leq \lfloor a_k/2 \rfloor$ ,  $x_1 \geq \lfloor a_k/2 \rfloor$  and  $1 \leq x_2 - x_1 \leq \lfloor a_k/2 \rfloor - \lfloor a_k/2 \rfloor \leq 0$ , which is a contradiction.

Therefore, we conclude in every case that  $a_2 \geq a_k - 1$  or  $G \in \mathcal{E}$ . Hence, Lemma 3.3 gives  $k = 2$ , or  $k = 3$  and  $a_2 = a_3 - 1$ , or  $k = 4$ ,  $a_2 = a_4 - 2$ ,  $a_3 = a_4 - 1$  and  $a_4$  is odd.  $\square$

We also have a sharp lower bound for the hyperbolicity constant.

**Theorem 3.8.** *For any integers  $k > 1$  and  $1 < a_2 < \dots < a_k$  we have*

$$\delta(C_\infty(1, a_2, \dots, a_k)) \geq \frac{3}{2},$$

and the equality is attained if  $a_k = k$ .

*Proof.* Define  $G = C_\infty(1, a_2, \dots, a_k)$ , and consider the geodesics  $\gamma_1, \gamma_2$  in  $G$  given by

$$\begin{aligned} \gamma_1 &:= [0, a_k] \cup [a_k, 2a_k] \cup [2a_k, 2a_k + 1], \\ \gamma_2 &:= [0, 1] \cup [1, 1 + a_k] \cup [1 + a_k, 1 + 2a_k]. \end{aligned}$$

Let  $T$  be the geodesic bigon  $T = \{\gamma_1, \gamma_2\}$  in  $G$ . If  $p$  is the midpoint of  $[a_k, 2a_k]$ , then

$$\delta(G) \geq d_G(p, \gamma_2) = \min \left\{ d_G(p, a_k) + d_G(a_k, 1 + a_k), d_G(p, 2a_k) + d_G(2a_k, 2a_k + 1), \frac{1}{2} L(\gamma_1) \right\} = \frac{3}{2}.$$

Assume now that  $a_k = k$ , i.e.,  $G = C_\infty(1, a_2, \dots, a_k) = C_\infty(1, 2, \dots, k)$ . Therefore,  $d_G(m, m + w) = 1$  for every  $m, w \in \mathbb{Z}$  with  $|w| \leq k$ . Note that if  $x, y \in J(G)$  and  $x$  and  $y$  are not related, then  $|x_1 - y_1| < k$ ,  $d_G(x_1, y_1) = 1$  and

$$d_G(x, y) \leq d_G(x, x_1) + d_G(x_1, y_1) + d_G(y_1, y) = 2. \tag{7}$$

Let us consider a geodesic triangle  $T = \{x, y, z\}$  in  $G$  and  $p \in [xy]$ ; by Theorem 3.2 we can assume that  $T$  is a cycle with  $x, y, z \in J(G)$ . We consider several cases.

(A) If  $x$  and  $y$  are not related, then (7) gives

$$d_G(p, [xz] \cup [yz]) \leq d_G(p, \{x, y\}) \leq \frac{1}{2} d_G(x, y) \leq 1.$$

(B) Assume that  $xRy$ . Without loss of generality we can assume that  $xLy$ . Define  $w_j, i_0, x_0$  and  $y_0$  as in the proof of Theorem 3.7. Then  $w_{i_0-1} < x_2 \leq w_{i_0} = x_0$ .

If  $x_0 = x_2$ , then  $L([xx_0]) \leq 1/2$ . If  $x_0 > x_2$ , then  $w_{i_0-1} < x_2 < w_{i_0}$ . Since  $x_2 - w_{i_0} \leq w_{i_0} - w_{i_0-1} \leq k$ ,  $d_G(x_2, x_0) = 1$  and

$$L([xx_0]) = d_G(x, x_0) \leq d_G(x, x_2) + d_G(x_2, x_0) \leq \frac{1}{2} + 1 = \frac{3}{2}.$$

Therefore, in both cases, if  $p \in [xx_0]$ , then

$$d_G(p, [xz] \cup [yz]) \leq d_G(p, x) \leq L([xx_0]) \leq \frac{3}{2}.$$

A similar argument to the previous one shows that if  $p \in [y_0y]$ , then  $d_G(p, [xz] \cup [yz]) \leq 3/2$ . If  $y_0 \in [xx_0]$ , then  $d_G(p, [xz] \cup [yz]) \leq 3/2$  holds for every  $p \in [xy]$ . Consider now the case  $y_0 \notin [xx_0]$ .

Since  $x_2 \leq x_0$  and  $y_0 \leq y_1$ , every  $p \in V(G) \cap [x_0y_0] \subset [xy]$  verifies  $x_2 \leq p \leq y_1$ . Since  $T$  is a continuous curve, we obtain

$$([xz] \cup [yz]) \cap \{x_1, x_2\} \neq \emptyset, \quad ([xz] \cup [yz]) \cap \{y_1, y_2\} \neq \emptyset.$$

Since  $[xz] \cup [yz]$  is a continuous curve joining  $x$  and  $y$ , if  $p \in V(G) \cap [x_0y_0]$ , then there exist  $u, v \in V(G) \cap ([xz] \cup [yz])$  with  $[u, v] \in E(G)$  and  $u \leq p \leq v$ . Hence,  $p - u \leq v - u \leq k$ ,  $v - p \leq v - u \leq k$  and

$$d_G(p, [xz] \cup [yz]) \leq d_G(p, \{u, v\}) = 1.$$

Therefore, if  $p \in [x_0y_0]$ , then

$$d_G(p, [xz] \cup [yz]) \leq \frac{3}{2}.$$

These inequalities give  $\delta(T) \leq 3/2$  and, hence,

$$\delta(C_\infty(1, 2, \dots, k)) \leq \frac{3}{2}.$$

Since we have proved the converse inequality, we conclude that the equality holds. □

As usual, the *complement*  $\overline{G}$  of the graph  $G$  is defined as the graph with  $V(\overline{G}) = V(G)$  and such that  $e \in E(\overline{G})$  if and only if  $e \notin E(G)$ . We are going to bound the hyperbolicity constant of the complement of every infinite circulant graph. In order to do it, we need some preliminaries.

For any graph  $G$ , we define,

$$\begin{aligned} \text{diam } V(G) &:= \sup \{d_G(v, w) \mid v, w \in V(G)\}, \\ \text{diam } G &:= \sup \{d_G(x, y) \mid x, y \in G\}. \end{aligned}$$

We need the following well-known result (see a proof, *e.g.*, in [36, Theorem 8]).

**Theorem 3.9.** *In any graph  $G$  the inequality  $\delta(G) \leq (\text{diam } G)/2$  holds.*

We have the following direct consequence.

**Corollary 3.10.** *In any graph  $G$  the inequality  $\delta(G) \leq (\text{diam } V(G) + 1)/2$  holds.*

From [34, Proposition 5 and Theorem 7] we deduce the following result.

**Lemma 3.11.** *Let  $G$  be any graph with a cycle  $g$ . If  $L(g) \geq 3$ , then  $\delta(G) \geq 3/4$ . If  $L(g) \geq 4$ , then  $\delta(G) \geq 1$ .*

We say that a vertex  $v$  of a graph  $G$  is a *cut-vertex* if  $G \setminus \{v\}$  is not connected. A graph is *two-connected* if it does not contain cut-vertices.

We need the following result in [26, Proposition 4.5 and Theorem 4.14].

**Theorem 3.12.** *Assume that  $G$  is a two-connected graph. Then  $G$  verifies  $\delta(G) = 1$  if and only if  $\text{diam } G = 2$ . Furthermore,  $\delta(G) \leq 1$  if and only if  $\text{diam } G \leq 2$ .*

**Definition 3.13.** *Let us consider integers  $k \geq 1$  and  $1 \leq a_1 < a_2 < \dots < a_k$ . We say that  $\{a_1, a_2, \dots, a_k\}$  is a 1-modulated sequence if for every  $x, y \in \mathbb{Z}$  with  $|y| \notin \{a_1, a_2, \dots, a_k\}$  we have  $|x| \notin \{a_1, a_2, \dots, a_k\}$  or  $|x - y| \notin \{a_1, a_2, \dots, a_k\}$ .*

We have the following sharp bounds for the hyperbolicity constant of the complement of every infinite circulant graph. In particular, they show that the complement of infinite circulant graphs are hyperbolic.

**Theorem 3.14.** *For any integers  $k \geq 1$  and  $1 \leq a_1 < a_2 < \dots < a_k$  we have*

$$1 \leq \delta(\overline{C_\infty(a_1, a_2, \dots, a_k)}) \leq \frac{3}{2}.$$

*Furthermore,  $\delta(\overline{C_\infty(a_1, a_2, \dots, a_k)}) = 1$  if and only if  $\{a_1, a_2, \dots, a_k\}$  is 1-modulated. If there is  $1 \leq j < a_k/5$  with  $j, 5j \notin \{a_1, a_2, \dots, a_k\}$  and  $2j, 3j, 4j \in \{a_1, a_2, \dots, a_k\}$ , then  $\delta(\overline{C_\infty(a_1, a_2, \dots, a_k)}) = 3/2$ .*

*Proof.* Define  $G := C_\infty(a_1, a_2, \dots, a_k)$ . Given  $u, v \in \mathbb{Z}$ , consider  $w \in \mathbb{Z}$  with  $w > u + a_k$  and  $w > v + a_k$ . Since  $[u, w], [v, w] \notin E(G)$ , we have  $[u, w], [v, w] \in E(\overline{G})$  and  $d_{\overline{G}}(u, v) \leq d_{\overline{G}}(u, w) + d_{\overline{G}}(w, v) = 2$ . Hence,  $\text{diam } V(\overline{G}) \leq 2$  and Corollary 3.10 gives  $\delta(\overline{G}) \leq 3/2$ .

Since  $[0, a_k + 1], [a_k + 1, 2a_k + 2], [2a_k + 2, 3a_k + 3], [3a_k + 3, 0] \notin E(G)$ , we have  $[0, a_k + 1], [a_k + 1, 2a_k + 2], [2a_k + 2, 3a_k + 3], [3a_k + 3, 0] \in E(\overline{G})$ . Since the cycle  $C := \{0, a_k + 1, 2a_k + 2, 3a_k + 3, 0\}$  in  $\overline{G}$  has length 4, Lemma 3.11 gives  $\delta(\overline{G}) \geq 1$ .

Since  $G$  is a circulant graph, the sequence  $\{a_1, a_2, \dots, a_k\}$  is 1-modulated if and only if for every  $x, y_1, y_2 \in \mathbb{Z}$  with  $|y_2 - y_1| \notin \{a_1, a_2, \dots, a_k\}$ , we have  $|x - y_1| \notin \{a_1, a_2, \dots, a_k\}$  or  $|x - y_2| \notin \{a_1, a_2, \dots, a_k\}$ . This happens if and only if  $d_{\overline{G}}(x, [y_1, y_2]) \leq 1$  for every  $x \in V(\overline{G})$  and  $[y_1, y_2] \in E(\overline{G})$ , and this condition is equivalent to  $\text{diam } V(\overline{G}) \leq 2$ . Since  $\overline{G}$  is a two-connected graph, Theorem 3.12 that  $\text{diam } \overline{G} \leq 2$  if and only if  $\delta(\overline{G}) \leq 1$ . Since  $\delta(\overline{G}) \geq 1$ , we conclude that  $\delta(\overline{G}) = 1$  if and only if  $\{a_1, a_2, \dots, a_k\}$  is 1-modulated.

Assume that there is  $1 \leq j < a_k/5$  with  $j, 5j \notin \{a_1, a_2, \dots, a_k\}$  and  $2j, 3j, 4j \in \{a_1, a_2, \dots, a_k\}$ , and consider the cycle  $C := \{0, j, 2j, 3j, 4j, 5j, 0\}$  in  $\overline{G}$  with length 6. Let  $x$  and  $y$  be the midpoints of the edges  $[2j, 3j]$  and  $[5j, 0]$ , respectively. Since  $d_{\overline{G}}([2j, 3j], [5j, 0]) = 2$ ,  $d_{\overline{G}}(x, y) = 3$  and  $C$  contains two geodesics  $g_1, g_2$  joining  $x$  and  $y$ , with  $g_1 \cap V(\overline{G}) = \{0, j, 2j\}$  and  $g_2 \cap V(\overline{G}) = \{3j, 4j, 5j\}$ . Since  $d_{\overline{G}}(j, \{3j, 4j, 5j\}) \geq 2$ , we have  $\delta(\overline{G}) \geq d_{\overline{G}}(j, g_2) = d_{\overline{G}}(j, \{x, y\}) = 3/2$ , and we conclude  $\delta(\overline{G}) = 3/2$ .  $\square$

Since Paley graphs is an important class of circulant graph, which is attracting great interest in recent years (see, e.g., [45]), we finish this paper with a result on the hyperbolicity constant of Paley graphs.

Recall that the Paley graph of order  $q$  with  $q$  a prime power is a graph on  $q$  nodes, where two nodes are adjacent if their difference is a square in the finite field  $GF(q)$ . This graph is circulant when  $q \equiv 1 \pmod{4}$ . Paley graphs are self-complementary, strongly regular, conference graphs, and Hamiltonian.

**Proposition 3.15.** *For any Paley graph  $G$  we have*

$$1 \leq \delta(G) \leq \frac{3}{2}.$$

*Proof.* Let us denote by  $n$  the cardinality of  $V(G)$ .

Since Paley graphs are self-complementary (the complement of any Paley graph is isomorphic to it), the degree of any vertex is  $(n - 1)/2$ . Hence, given  $u, v \in V(G)$  with  $[u, v] \notin E(G)$ , there exists a vertex  $w \in V(G)$  with  $d_G(u, w) = d_G(u, v) = 1$ , and we conclude that  $\text{diam } V(G) \leq 2$ . Therefore, Corollary 3.10 gives  $\delta(G) \leq 3/2$ .

Since  $G$  is a Hamiltonian graph, there exists a Hamiltonian cycle  $g$ . Since  $L(g) = n \geq 5$ , Lemma 3.11 gives  $\delta(G) \geq 1$ .  $\square$

**Acknowledgement:** The authors thank the referees for their deep revision of the manuscript. Their comments and suggestions have contributed to improve substantially the presentation of this work. This work is supported in part by two grants from Ministerio de Economía y Competitividad (MTM2013-46374-P and MTM2015-69323-REDT), Spain, and a grant from CONACYT (FOMIX-CONACYT-UAGro 249818), México.

The first author is supported in part by two grants from Ministerio de Economía y Competitividad (MTM2013-46374-P and MTM2015-69323-REDT), Spain, and a grant from CONACYT (FOMIX-CONACYT-UAGro 249818), México.

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