

## Generating Function: Multiple Orthogonal Polynomials

W. Carballosa

Universidad Autónoma de Zacatecas  
Paseo la Bufa, int. Calzada Solidaridad, 98060, Zacatecas, Mexico

J. C. Hernández-Gómez

Universidad Autónoma de Guerrero  
Carlos E. Adame No. 54 Col. Garita, 39650, Acalpulco Gro., Mexico

L. R. Piñeiro

Universidad de la Habana  
San Lázaro y L. Municipio Plaza de la Revolución, La Habana, Cuba

José M. Sigarreta

Universidad Autónoma de Guerrero  
Carlos E. Adame No. 54 Col. Garita, 39650, Acalpulco Gro., Mexico

Copyright © 2015 W. Carballosa, J. C. Hernández-Gómez, L. R. Piñeiro and José M. Sigarreta. This article is distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

### Abstract

In this paper we present a general methodology to obtain a generating function for some multiple orthogonal polynomials of type I with regular indices. In particular, we obtain an explicit generating functions  $P_x(t) := \sum_{n=0}^{\infty} Q_{2n}(x)t^n$  and  $I_x(t) := \sum_{n=0}^{\infty} Q_{2n+1}(x)t^n$  with  $Q_n(x) = q_{n,1}(x) + x q_{n,2}(x)$  where  $(q_{n,1}(x), q_{n,2}(x))$  is the  $r$ -vector of type I associated with the multiple orthogonal Hermite polynomial with regular index for  $r = 2$ .

**Mathematics Subject Classification:** 33C45, 42C05

**Keywords:** Methodology, Generating Function, Orthogonal Polynomials

## 1 Introduction

At present, interest in the study of multiple orthogonal polynomials has increased due to the development of simultaneous rational approximation of functions and the link between the two theories. Orthogonal polynomials and Padé approximants are essential areas for research, because of its applications in different branches of mathematics such as number theory, the problem of moments, the analytic extension, interpolation problems, spectral theory of operators and others.

Initial ideas of the theory of simultaneous approximation is found in the works of Chebyshev and Markov. The construction of simultaneous rational approximation was introduced by Hermite in 1873 during the demonstration of the importance of the Euler number [5], so it is known by the name of Hermite-Padé approximants (H-P). In [7] the author obtained generating functions for the family of multiple orthogonal polynomials of Laguerre and Hermite of type II from the Rodrigues formulae.

## 2 Hermite-Padé Approximants.

Let  $f = (f_1, \dots, f_r)$  be a set of  $r$  analytic functions in a neighborhood of infinity,

$$f_i(z) \approx \sum_{n \geq 0} \frac{c_{n,i}}{z^{n+1}}, \quad \text{with } c_{n,i} \in \mathbb{C} \quad \text{and } i = 1, \dots, r. \quad (1)$$

One  $k$ -dimensional multi-index  $\vec{n}$  is a (non-negative) integer vector  $\vec{n} = (n_1, \dots, n_k)$  for some  $k \in \mathbb{N}^*$ . Denoted by  $n := \|\vec{n}\|_1 = \sum_{j=1}^k n_j$ .

One way to extend the rational approximation of a function is through formulation of the following classic problem.

**Problem 1.** *Given a finite set of  $r$  analytic functions  $f_1, \dots, f_r$  in a neighborhood of infinity and the multi-index  $\vec{n} = (n_1, \dots, n_r)$ , search a non null polynomial  $Q_{\vec{n}}$  of degree not greater than  $n$ , and polynomials  $P_{(\vec{n},1)}, \dots, P_{(\vec{n},r)}$ , such that for each  $j$  with  $j = 1, \dots, r$  we have that*

$$Q_{\vec{n}}(z)f_j(z) - P_{(\vec{n},j)}(z) = O\left(\frac{1}{z^{n_j+1}}\right). \quad (2)$$

It is known that this problem always has a solution, the homogeneous system of linear equations determines the associated polynomial  $Q_{\vec{n}}$ , then each equation determines  $P_{(\vec{n},j)}$ . The Hermite-Padé approximants of type II can be defined as follows.

**Definition 2.1.** Let  $(Q_{\vec{n}}, P_{(\vec{n},1)}, \dots, P_{(\vec{n},r)})$  be a solution of Problem 1. The  $r$ -vector  $\pi_{\vec{n}} = (\pi_{(\vec{n},1)}, \dots, \pi_{(\vec{n},r)})$  of rational fractions of the following form

$$\pi_{(\vec{n},j)}(z) = \frac{P_{(\vec{n},j)}(z)}{Q_{\vec{n}}(z)}, \quad \text{for every } 1 \leq j \leq r, \tag{3}$$

are called Hermite-Padé approximants of type II in the point  $z = \infty$  with multi-index  $\vec{n} = (n_1, \dots, n_r)$ .

The existence of Hermite-Padé approximants of type II (solution of Problem 1) has been resolved. However, in general, the fractions  $\{\pi_{(\vec{n},i)}(z)\}_{i=1}^r$  are not uniquely defined given the multi-index and the  $r$  analytic functions at infinity, so it is useful to establish sufficient conditions to ensure uniqueness.

**Definition 2.2.** A multi-index  $\vec{n}$  is called normal index for Problem 1, if in every solution  $(Q_{\vec{n}}, P_{(\vec{n},1)}, \dots, P_{(\vec{n},r)})$ , the polynomial  $Q_{\vec{n}}(z)$  has a degree of  $n$ .

If  $\vec{n}$  is a normal index, then the uniqueness of the Hermite-Padé approximants is guaranteed. Another way to extend the rational approximation of a function is given by the following problem.

**Problem 2.** Given a set of  $r$  analytic functions  $f_1, \dots, f_r$  in a neighborhood of infinity and the multi-index  $\vec{n} = (n_1, \dots, n_r)$ , search for a  $r$ -vector of polynomials  $(A_{\vec{n},1}, \dots, A_{\vec{n},r})$ , with  $A_{\vec{n},j}$  of degree not greater than  $n_j - 1$  and one polynomial  $B_{\vec{n}}$  such that

$$\sum_{j=1}^r A_{\vec{n},j}(z)f_j(z) - B_{\vec{n}}(z) = O\left(\frac{1}{z^n}\right). \tag{4}$$

In order to solve Problem 2, the associated homogeneous system of linear equations that allows us to determine the polynomials  $A_{\vec{n},j}$  is solved and subsequently is obtained  $B_{\vec{n}}$ . The Hermite-Padé approximants of type I can be defined as follows.

**Definition 2.3.** Let  $(A_{\vec{n},1}, \dots, A_{\vec{n},r}, B_{\vec{n}})$  be a solution of Problem 2. The linear combination with polynomial coefficients of the  $r$  functions

$$\sum_{j=1}^r A_{n,j}(z)f_j(z) \approx B_n(z), \tag{5}$$

are called Hermite-Padé approximants of type I in the point  $z = \infty$  with multi-index  $\vec{n} = (n_1, \dots, n_r)$ .

**Definition 2.4.** A multi-index  $\vec{n} = (n_1, \dots, n_r)$ , is a normal index for the Problem 2, if in any solution  $(A_{\vec{n},1}, \dots, A_{\vec{n},r}, B_{\vec{n}})$ , the polynomial  $A_{\vec{n},j}(z)$  has degree  $n_j - 1$ .

The existence of Hermite-Padé approximants of type I (solution of Problem 2) is solved, but new conditions are needed to ensure uniqueness.

Multiple orthogonal polynomials of types I and II are directly related to the Hermite-Padé approximants of types I and II, respectively, associated with  $r$  Markov's functions <sup>1</sup>. Also these fulfill interesting orthogonality conditions which satisfy each family of multiple orthogonal polynomials. In this paper, we work with  $r$  Lebesgue measures  $\mu_1, \dots, \mu_r$  with supports  $\text{supp}(\mu_1), \dots, \text{supp}(\mu_r)$ , respectively, all of them formed by an infinite number of points contained in closed subsets  $E_1, \dots, E_r$  in the real axis (*i.e.*,  $\text{supp}(\mu_i) \subset E_i \subset \mathbb{R}$  for  $1 \leq i \leq r$ ). If  $E_i$  is an unbounded set for  $1 \leq i \leq r$ , it is further assumed that  $x^n \in \mathcal{L}_1[\mu_i]$  for all  $n \in \mathbb{Z}_+$ . In an alternative way, working with  $r$  weight functions  $w_1, \dots, w_r$  associated with the above measures.

**Definition 2.5.** A  $r$ -vector  $(A_{\vec{n},1}, \dots, A_{\vec{n},r})$  is a multiple orthogonal polynomial of type I associated to the weight functions  $w_1, \dots, w_r$  for the multi-index  $\vec{n} = (n_1, \dots, n_r)$ , if each  $A_{\vec{n},j}$  has degree less than  $n_j$  and the orthogonality condition is satisfied

$$\sum_{j=1}^r \int_{E_j} x^k A_{\vec{n},j}(x) w_j(x) dx = 0 \quad \text{for } k = 0, 1, \dots, n - 2. \quad (6)$$

**Definition 2.6** (Multiple orthogonal polynomial of type II). A polynomial  $P_{\vec{n}}$  is a multiple orthogonal Polynomial of type II associated with the weight function  $w_1, \dots, w_r$  for the multi-index  $\vec{n} = (n_1, \dots, n_r)$ , if has degree  $n$  and the orthogonality conditions are satisfied

$$\int_{E_j} P_{\vec{n}}(x) w_j(x) x^k dx = 0 \quad \text{for } k = 0, 1, \dots, n_j - 1 \quad \text{with } j = 1, \dots, r. \quad (7)$$

There is a well known equivalence between determining multiple orthogonal polynomials and Hermite-Padé approximants of types I and II, see [9, 10, 11]. To see how to generate the multiple orthogonal polynomials of types I and II, and the associated polynomials of the second kind, see [2, 3, 4, 8, 10, 11].

<sup>1</sup>Markov's functions are particular cases of functions satisfying (1). Depending on the nature of the support of the measurement of the orthogonality problem associated also are called Stieltjes functions

Working with multi-index, in general way, in the study of multiple orthogonal polynomials becomes quite cumbersome. In this direction we will work with a regular multi-index.

**Definition 2.7.** A multi-index  $\vec{n} = (n_1, \dots, n_r)$  is a regular index<sup>2</sup> if for  $1 \leq i < j \leq r$  we have  $0 \leq n_i - n_j \leq 1$ .

This definition allows to associate for all  $n \in \mathbb{N}$  an unique  $r$ -dimensional regular index  $\vec{m}$  such that  $n = \|\vec{m}\|_1$ .

**Definition 2.8.** A system of analytic functions  $f_1, \dots, f_r$  is weakly perfect if all regular indices are normal.

By assuming that the multi-indexes are regular we adopt the subscript notation  $n(X_n)$  to denote the  $r$ -dimensional multi-index  $\vec{m}$  which satisfies  $\|\vec{m}\|_1 = n$  and will be denoted by  $n_i$  to the  $i$ -th component of this multi-index. In addition, we will work with families of classics multiple orthogonal polynomials according to the classification in [1] given by weight functions which are solution of the Pearson's equation in Angelesco Systems and AT-systems. We also assume the results in a weakly perfect system where multiple orthogonal type II polynomials are monic.

**Theorem 2.9** (Recurrence Formulae). Let  $\{P_n(x)\}_{n=0}^\infty$  be a family of multiple orthogonal monic polynomials of type II associated to the Lebesgue measures  $\mu_1, \dots, \mu_r$  of regular index in a weakly perfect system. Then

$$xP_n(x) = P_{n+1}(x) + \sum_{j=0}^r a_{n,j} P_{n-j}(x), \quad n \geq 0, \tag{8}$$

with initial conditions  $P_0(x) = 1$  and  $P_{-k}(x) = 0$  for  $k = 1, \dots, r$ .

The proof of the Theorem 2.9 is obtained using (7) with the help of the bilinear form (9), extended from the moments problem of orthogonal polynomials which guarantees the existence of biorthogonal families  $\{P_n\}_{n=0}^\infty$  and  $\{Q_n\}_{n=0}^\infty$  according to (9) in weakly perfect systems.

$$\langle p(x), q(x) \rangle = \sum_{i=1}^r \int_{E_i} p(x)q_i(x)d\mu_i(x) \quad \text{with} \quad q(x) = \sum_{i=1}^r x^{i-1}q_i(x^r). \tag{9}$$

This bilinear form is well defined due to the existence and uniqueness of the decomposition for  $q(x)$ .

---

<sup>2</sup>regular index has the form  $\left( \frac{k+1, \dots, k+1}{j \text{ times}}, \frac{k, \dots, k}{r-j \text{ times}} \right)$ .

**Theorem 2.10.** *Let  $\{(q_{n,1}, \dots, q_{n,r})\}_{n=0}^{\infty}$  be a family of  $r$ -vectors multiple orthonormal polynomials of type I associated to the Lebesgue measures  $\mu_1, \dots, \mu_r$  of regular index in a weakly perfect system. Then*

$$xq_{n,i} = q_{n-1,i} + \sum_{j=0}^r a_{n+j,j} q_{n+j,i}(x), \quad \text{for } n \geq 0 \quad \text{and } i = 1, \dots, r, \quad (10)$$

for the coefficients  $a_{n,j}$  used in Theorem 2.9.

The proof of Theorem 2.10 is obtained using (6) and using the bilinear form (9). By the Theorem 2.10 a recurrence relation with  $r + 2$  terms that satisfy the family of polynomials  $\{Q_n\}_{n=0}^{\infty}$ .

$$x^r Q_n(x) = Q_{n-1}(x) + \sum_{j=0}^r a_{n+j,j} Q_{n+j}(x) \quad \text{for } n \geq 0, \quad (11)$$

for the coefficients  $a_{n,j}$  used in the Theorem 2.9.

The decomposition of  $Q_n$  in the  $n$ -th  $r$ -vector multiple orthogonal polynomial of type I, we will call to the polynomials  $Q_n(x) = \sum_{i=1}^r x^{i-1} q_{n,i}(x^r)$  *multiple orthogonal polynomials of type I* and to establish differences,  $(q_{n,1}, \dots, q_{n,r})$  will be referred by the expression  *$r$ -vector multiple orthogonal of type I*.

### 3 Generating function and multiple orthogonal Hermite polynomials of type I.

The normality of the regular index is guaranteed for AT-systems and Angelesco systems, see [10]. In [10] the authors call family of classic multiple orthogonal polynomials to three stated families for Angelesco systems (Jacobi-Angelesco, Jacobi-Laguerre and Laguerre-Hermite) and four families in AT-systems (multiple orthogonal Laguerre I, multiple orthogonal Laguerre II and multiple orthogonal Hermite). Our work focuses, for example, on the family multiple orthogonal Hermite polynomials; one of the four associated with the AT-systems. For more information on the multiple orthogonal polynomials on the real line; see, for instance, [1, 6, 10, 11].

Multiple orthogonal Hermite polynomials  $H_n^c$ , have support on the real line  $(-\infty, \infty)$  and weight functions are given by  $\{w_j(x) = e^{-x^2+c_jx}\}_{j=1}^r$  for different real numbers  $c_j$ . Multiple orthogonal polynomials of Hermite for  $r = 2$  ( $H_n^{(c_1, c_2)}$ ) satisfy the orthogonality condition.

$$\int_{\mathbb{R}} H_n^{(c_1, c_2)}(x) e^{-x^2+c_i x} x^k dx = 0 \quad \text{for } k = 0, \dots, n_i - 1, \quad \text{and } i = 1, 2. \quad (12)$$

On the orthogonality relation and the definition of their weight functions, these families of polynomials can be obtained as the limit from the Jacobi-Piñeiro, for the particular case  $r = 2$  is as follows, see [10].

$$H_n^{(c_1, c_2)}(x) = \lim_{\alpha \rightarrow \infty} 2(\sqrt{\alpha})^n P_n^{(\alpha, \alpha + c_1\sqrt{\alpha}, \alpha + c_2\sqrt{\alpha})} \left( \frac{x + \sqrt{\alpha}}{2\sqrt{\alpha}} \right). \tag{13}$$

From computations in [10] we can obtain:

$$\begin{aligned} a_{2n,0} &= \frac{c_1}{2}, & a_{2n+1,0} &= \frac{c_2}{2}, & a_{n,1} &= \frac{n}{2}, & a_{2n,2} &= \frac{n(c_1 - c_2)}{4}, \\ a_{2n+1,2} &= \frac{n(c_2 - c_1)}{4}. \end{aligned} \tag{14}$$

These results allow us to easily generate multiple orthogonal polynomials of Hermite by their respective recurrence formulae. Note that the recurrence coefficients of same parity are given by identical expressions, which suggests to take as generating functions to  $P_x(t) = \sum_{n=0}^{\infty} Q_{2n}(x)t^n$  and  $I_x(t) = \sum_{n=0}^{\infty} Q_{2n+1}(x)t^n$ .

Assuming that you are working with power series of  $t$  in a non-empty disc of convergence  $\mathcal{D}$ , so far undetermined; therefore  $P_x(t)$  and  $I_x(t)$  are defined correctly in  $\mathcal{D}$ .

**Theorem 3.1.** *Let  $\{Q_n\}_{n=0}^{\infty}$  be a family of multiple orthogonal polynomials of Hermite of type I associated with weight functions  $w_1(x) = e^{-x^2+c_1x}$  and  $w_2(x) = e^{-x^2+c_2x}$  with  $c_1 \neq c_2$ . Consider the generating functions  $P_x(t) = \sum_{n=0}^{\infty} Q_{2n}(x)t^n$  and  $I_x(t) = \sum_{n=0}^{\infty} Q_{2n+1}(x)t^n$ . Then, we have*

$$\begin{aligned} P_x(t) &= \frac{e^{-t}}{\left(1 + \frac{t}{a^2}\right)^{\frac{1}{4}}} \left\{ \left[ (Q_0(x) - aQ_1(x)) \left(aB + \frac{1}{4}\right) f(t) + BtQ_1(x) \right] \right. \\ &\quad \left. \frac{\sinh\left(\frac{\sqrt{D}}{4}\right)}{\sqrt{D}/4} + Q_0(x) \cosh\left(\frac{\sqrt{D}}{4}\right) \right\} \end{aligned} \tag{15}$$

and

$$\begin{aligned} I_x(t) &= \frac{e^{-t}}{\left(1 + \frac{t}{a^2}\right)^{\frac{1}{4}}} \left\{ \left[ BQ_0(x) - \left(aB + \frac{1}{4}\right) Q_1(x) \right] \right. \\ &\quad \left. f(t) \frac{\sinh\left(\frac{\sqrt{D}}{4}\right)}{\sqrt{D}/4} + Q_1(x) \cosh\left(\frac{\sqrt{D}}{4}\right) \right\}. \end{aligned} \tag{16}$$

*Proof.* Let  $n$  be a fixed real number. Multiplying (10) by  $t^n$  and then summing these equations, separated by parity of  $n$ , we obtain (17) and (18). By uniform convergence on every compact in  $\mathcal{D}$ , the functions  $P_x'(t)$  and  $I_x'(t)$  are well defined on  $\mathcal{D}$ .

$$\frac{c_1 - c_2}{4} P_x'(t) + t I_x'(t) = \left(x^2 - \frac{c_1}{2}\right) P_x(t) - \left(t + \frac{1}{2}\right) I_x(t). \quad (17)$$

$$P_x'(t) - \frac{c_1 - c_2}{4} I_x'(t) = -P_x(t) + \left(x^2 - \frac{c_2}{2}\right) I_x(t). \quad (18)$$

From these equations the following problem is obtained

$$\begin{pmatrix} \frac{c_1 - c_2}{4} & t \\ 1 & -\frac{c_1 - c_2}{4} \end{pmatrix} \begin{pmatrix} P_x'(t) \\ I_x'(t) \end{pmatrix} = \begin{pmatrix} x^2 - \frac{c_1}{2} & -(t + \frac{1}{2}) \\ -1 & x^2 - \frac{c_2}{2} \end{pmatrix} \begin{pmatrix} P_x(t) \\ I_x(t) \end{pmatrix}, \quad (19)$$

$$\text{with } \begin{pmatrix} P_x(0) \\ I_x(0) \end{pmatrix} = \begin{pmatrix} Q_0(x) \\ Q_1(x) \end{pmatrix}.$$

We have

$$\begin{pmatrix} -a & -t \\ -1 & a \end{pmatrix} \begin{pmatrix} a & t \\ 1 & -a \end{pmatrix} = -(t + a^2) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

taking  $a = \frac{c_1 - c_2}{4}$ ,  $b = x^2 - \frac{c_1}{2}$ ,  $c = x^2 - \frac{c_2}{2}$  and  $B := a + b = c - a$ , we obtain for  $t \neq -a^2$  the system of equations (19) is equivalent to the following Cauchy problem.

$$\begin{pmatrix} P_x'(t) \\ I_x'(t) \end{pmatrix} = \frac{1}{t + a^2} \begin{pmatrix} -t + ab & Bt - \frac{a}{2} \\ B & -t - (ac + \frac{1}{2}) \end{pmatrix} \begin{pmatrix} P_x(t) \\ I_x(t) \end{pmatrix}, \quad (20)$$

$$\text{with } \begin{pmatrix} P_x(0) \\ I_x(0) \end{pmatrix} = \begin{pmatrix} Q_0(x) \\ Q_1(x) \end{pmatrix}.$$

Since (20) is a system of homogeneous linear differential equations having the form  $X' = A(t)X$  given  $X(0)$ , its solution is given by the quadrature formulas  $X = \exp\left(\int_0^t A(\tau) d\tau\right) X(0)$ . Note that the largest domain of (20) is the open ball  $B_{a^2}(0)$  with center at 0 and its radius  $a^2 > 0$ . Besides,



$$\begin{aligned}
 M = \int_0^t A(\tau) d\tau &= \begin{pmatrix} \int_0^t \frac{ab-\tau}{\tau+a^2} d\tau & \int_0^t \frac{B\tau-\frac{a}{2}}{\tau+a^2} d\tau \\ \int_0^t \frac{B}{\tau+a^2} d\tau & -\int_0^t \frac{\tau+ac-\frac{1}{2}}{\tau+a^2} d\tau \end{pmatrix} \\
 &= \begin{pmatrix} -t + aBf(t) & Bt - a\left(aB + \frac{1}{2}\right) \\ Bf(t) & -t - \left(aB + \frac{1}{2}\right) f(t) \end{pmatrix},
 \end{aligned}$$

where  $f(t) := \int_0^t \frac{d\tau}{\tau+a^2} = \sum_{n \geq 0} \frac{(-1)^n}{n+1} \left(\frac{t}{a^2}\right)^{n+1}$  that is,  $f$  is a function of Markov kind evaluated at  $-a^2$ . Note that the function  $f(t)$  is strictly increasing in  $-a^2 < t < a^2$ , nonnegative on  $0 \leq t < a^2$  with  $f(0) = 0$  and  $f(a^2) < \infty$ .

In order to determine the explicit expression of the generating functions,  $e^M$  is calculated through the Jordan matrix of  $M$ .

$$P_M(\lambda) = \lambda^2 - \text{Tr}(M)\lambda + \det(M) = \lambda^2 - \left(2t + \frac{f(t)}{2}\right) \cdot \lambda + t^2 - \left(B^2 - \frac{1}{2}\right) t f(t).$$

Therefore, the roots of  $P_M(\lambda)$  (eigenvalues of  $M$ ) are:

$$\lambda_i = \frac{-2t - \frac{f(t)}{2} + \frac{(-1)^{i-1}}{2} \sqrt{f^2(t) + 16B^2 t f(t)}}{2}, \quad \text{for } i = 1, 2. \quad (21)$$

Denote by  $D$  to discriminant of  $P_M(\lambda)$ , i.e.,  $D = f^2(t) + 16B^2 t f(t)$ . As  $B$  depend on  $x^2$  linearly and  $f(t) \leq f(a^2) < \infty$  for  $-a^2 < t \leq a^2$ , can be taken sufficiently large values of  $x^2$  such that, if  $t \neq 0$  we have  $\frac{f(t)}{t} + 16B^2 > 0$ . In this way we obtain  $\lambda_1 \neq \lambda_2$  for some domain of  $x$ .

To determine the associated eigenvector, the systems  $(M - \lambda_i I)V_i = 0$  for  $i = 1, 2$  are solved, resulting in

$$V_{1,2} = \begin{pmatrix} \left(aB + \frac{1}{4}\right) f(t) \pm \frac{\sqrt{D}}{4} \\ Bf(t) \end{pmatrix},$$

where  $\sqrt{D}$  represents the root which has positive real part to ensure continuity and differentiability. Then  $M = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}$ , where

$$P = \begin{pmatrix} \left(aB + \frac{1}{4}\right) f(t) + \frac{\sqrt{D}}{4} & \left(aB + \frac{1}{4}\right) f(t) - \frac{\sqrt{D}}{4} \\ Bf(t) & Bf(t) \end{pmatrix},$$

and

$$P^{-1} = \frac{1}{2B\frac{\sqrt{D}}{4}f(t)} \begin{pmatrix} Bf(t) & -(aB + \frac{1}{4})f(t) + \frac{\sqrt{D}}{4} \\ Bf(t) & (aB + \frac{1}{4})f(t) + \frac{\sqrt{D}}{4} \end{pmatrix}.$$

Carrying out the matrix products to solve (20) we get

$$X = \frac{e^{-t}}{(1 + \frac{t}{a^2})^{\frac{1}{4}}} \left\{ \begin{pmatrix} (aB + \frac{1}{4})f(t) & (Bt - a(aB + \frac{1}{4})f(t)) \\ Bf(t) & -(aB + \frac{1}{4})f(t) \end{pmatrix} \right. \\ \left. \frac{\sinh(\sqrt{D}/4)}{\sqrt{D}/4} + I \cosh\left(\frac{\sqrt{D}}{4}\right) \right\} \begin{pmatrix} Q_0(x) \\ Q_1(x) \end{pmatrix} \quad (22)$$

Then verifying the initial conditions and getting the result is straightforward.  $\square$

**Proposition 1.** *The development of  $D$  in power series of  $t$  has convergency radius  $a^2$ .*

*Proof.* Since  $f(-a^2) = -\infty$ , we have immediately that the convergency radius of  $D$  is less than or equal to  $a^2$ . Finally we have the result, as  $D$  is combination of elementary functions of  $f$  and the radius of convergence of  $f$  is  $a^2$ .  $\square$

**Proposition 2.**  *$P_x(t)$  and  $I_x(t)$  as functions of  $t$  are analytic on the open ball  $B_{a^2}(0)$  centered at 0 with radius  $a^2$ . In addition, both have a non-algebraic singularity at  $t = -a^2$ .*

*Proof.* From Theorem 3.1, it can be verified that  $P_x(t)$  and  $I_x(t)$  have an algebraic singularity at  $t = -a^2$  due to the denominator of the first factor and further has a non-algebraic singularity at the same point because of the Markov function. Others possible singularities may arise due to the fraction  $\frac{\sinh(\sqrt{D}/4)}{\sqrt{D}/4}$ , but we can see that it is not in this way. Note that

$$\frac{\sinh(\sqrt{D}/4)}{\sqrt{D}/4} = \sum_{n \geq 0} \frac{(\sqrt{D}/4)^{2n}}{(2n+1)!}.$$

Therefore  $\frac{\sinh(\sqrt{D}/4)}{\sqrt{D}/4}$  is analytic in the region where  $\sqrt{D}/4$  is analytic too. Finally, by Proposition 1, the proof is concluded.  $\square$

## Conclusions

In this work, we present a general methodology to obtain a generating function for some multiple orthogonal polynomials of type I with regular indices. The methodology can be used on families of classical multiple orthogonal polynomials in AT-systems for which the recurrence coefficients are known and their expressions can be transformed in an appropriate way. For example, by analyzing the structure of the expression of the families of multiple orthogonal Laguerre polynomials I and II relative to the coefficients of recurrence. These families should be natural candidates to apply the methodology in order to obtain generating functions for its multiple orthogonal polynomials of type I. In this direction, we invite the reader to develop the proposed idea.

**Acknowledgements.** This paper was supported in part by a grant from Ministerio de Economía y Competitividad (MTM 2013-46374-P), Spain, and a grant from CONACYT (FOMIX-CONACyT-UAGro 249818), Mexico.

## References

- [1] A. I. Aptekarev, A. Branquinho and W. Van Assche, Multiple orthogonal polynomials for classical weights, *Trans. Amer. Math. Soc.*, **355** (2003), no. 10, 3887-3914. <http://dx.doi.org/10.1090/s0002-9947-03-03330-0>
- [2] B. Beckermann, J. Coussement and W. Van Assche, Multiple Wilson and Jacobi-Piñeiro polynomials, *Journal of Approximation Theory*, **132** (2005), 155-181. <http://dx.doi.org/10.1016/j.jat.2004.12.001>
- [3] W. Carballosa and L.R. Piñeiro, Classical Multiple orthogonal polynomials on the real line, *Cienc. Mat. (Havana)*, **25** (2009-2011), no. Unique, 73-91.
- [4] A. Gago-Alonso, L.R. Piñeiro-Díaz and L. Santiago-Moreno, Direct and inverse problems for the generalized relativistic Toda lattice and the connection with general orthogonal polynomials, *Inverse Problems*, **24** (2008), no. 2, 025009. <http://dx.doi.org/10.1088/0266-5611/24/2/025009>
- [5] C. Hermite, *Sur la Fonction Exponentielle*, Gauthier-Villars, 1874.
- [6] K. A. Kalyagin, A. Ronveaux, On an system of "classical" polynomials of simultaneous orthogonality, *J. Comput. Appl. Math.*, **67** (1996), 207-217. [http://dx.doi.org/10.1016/0377-0427\(94\)00129-4](http://dx.doi.org/10.1016/0377-0427(94)00129-4)

- [7] D. W. Lee, Properties of multiple Hermite and multiple Laguerre polynomials by the generating function, *Integral Transforms and Special Functions*, **18** (2007), no. 12, 855-869.  
<http://dx.doi.org/10.1080/10652460701510725>
- [8] L. R. Piñeiro, On simultaneous Padé approximants for a collection of Markov functions, *Vestnik Mosk Univ., Ser. I* (1987), no. 2, 67-70 (in Russian), translated in *Moscow Univ. Math. Bull.*, **42** (1987), 52-55.
- [9] V. N. Sorokin, Simultaneous Padé approximants for finite intervals, *Izv. Vyssh. Uchebn. Zaved. Mat.*, **8** (1984), 45-52.
- [10] W. Van Assche, Els Coussement, Some classical multiple orthogonal polynomials, *J. Comput. Appl. Math.*, **127** (2001), 317-347.  
[http://dx.doi.org/10.1016/s0377-0427\(00\)00503-3](http://dx.doi.org/10.1016/s0377-0427(00)00503-3)
- [11] W. Van Assche, Padé and Hermite-Padé Approximation and Orthogonality, *Surveys in Approximation Theory*, **2** (2006), 61-91.

**Received: November 8, 2015; Published: March 9, 2016**